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A Two-grid Discretization Scheme of Discontinuous Galerkin Method for the Steklov Eigenvalue Problem

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ABSTRACT: *The purpose of this article is to propose numerical solution and analyze the discontinuous Galerkin finite element methods of the Steklov eigenvalue problem. We provide a two-grid discretization scheme of discontinuous Galerkin method based on the shifted-inverse iteration. With the scheme, the solution of the Steklov eigenvalue problem on a fine grid is reduced to the solution of the Steklov eigenvalue problem on a much coarser grid and the solution of a linear algebraic system on the fine grid. Numerical results are provided to validate our theoretical findings.*

KEYWORDS - *Steklov eigenvalue problem, discontinuous Galerkin method, two-grid discretization, shifted-inverse iteration.*

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I. INTRODUCTION

Steklov eigenvalue problems in which the eigenvalue parameter appears in the boundary condition, arise in a number of applications. For instance, such spectral problems are found in the study of surface waves [1], in the stability of mechanical oscillators immersed in a viscous fluid [2], and in the vibration modes of a structure in contact with an incompressible fluid [3]. In recent years, more and more scholars studied the numerical methods for Steklov eigenvalue problems [4-9].

Finite element methods are the most commonly used numerical methods for solving eigenvalue problems. However, The first introduction of the discontinuous Galerkin(DG) method was analyzed in Reed and Hill for the approximation of linear hyperbolic problems. The main feature of DG method is that the test functions are discontinuous along the edges (or faces) of the mesh. DG method enjoys the following advantages such as local mass conservation, combination coupled with other methods easily, hp-adaptivity, working on polygonal meshes. Consequently, DG methods have been developed for many problems, and we cite as a minimal sample [10-13]. Moreover, DG methods for the eigenvalue problems have been discussed in many papers, for example, Laplacian eigenvalue problem [14], Steklov eigenvalue problem [15], biharmonic eigenvalue problem [16-17], transmission eigenvalue problem [18-19] and Maxwell eigenvalue problem [20-21].

In recent years, DGFEM have gained much interest due to their ease of treatment of highly unstructured meshes and inhomogeneous boundary conditions. The DGFEM have been attractive due to their flexibility in handling general meshes, non-uniformity in degree of approximation and capturing the rough solutions more accurately. Hence, the DGFEM has been developed and applied to solve various problems, for example, elliptic problems [22-23], hyperbolic problems [24-26], Navier-Stokes equations [27-28], etc.

In this work, we will further study the symmetric interior penalty discontinuous Galerkin methods (SIPG) for the Steklov eigenvalue problem. The main difficulty of the theoretical analysis stems from the complexity of bilinear forms of the SIPG method and its nonconformity. For addressing this problem, we rewrite the SIPG method in a discontinuous way by introducing a lifting operator, and then decompose the error into a conforming and nonconforming parts that are estimated separately. Note that these techniques have been applied for source problems [29] and Laplace eigenvalue problems [30].

Based on the above work, the remaining part of our article is arranged as follows: In Section 2, we first introduce the model problem and then describe the SIPG method and its error estimates. In Section 3, establish a two-grid discretization scheme of DGFEM based on the shifted-inverse iteration and the optimal convergence for the proposed scheme. In Section 4, conduct a theoretical analysis, that is to say, a priori error estimates of DGFEM for the source problem and the eigenvalue problem are presented. Finally, some numerical experimental results demonstrating our theoretical results are provided in Section 5.

Let $H^t(\Omega)$ and $H^t(\partial\Omega)$ denote Sobolev spaces on Ω and $\partial\Omega$ with real order t, respectively. The norm in $H^t(\Omega)$ and $H^t(\partial\Omega)$ are denoted by $\|\cdot\|_s$ and $\|\cdot\|_{s,\partial\Omega}$, respectively. $H^0(\Omega) = L^2(\Omega)$.

In this paper, we will write $a \lesssim b$ to indicate that $a \leq Cb$ with C being a generic positive constant independent of the mesh diameter which may take different values in different contexts, and write $a \gtrsim b$ when $a \geq Cb$ for some positive constant C for simplicity.

II. PRELIMINARIES

We consider the following Steklov eigenvalue problem:

$$-\Delta u + u = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial \mathbf{n}} = \lambda u \text{ on } \partial\Omega \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain with Lipschitz continuous boundary $\partial\Omega$, $\frac{\partial u}{\partial \mathbf{n}}$ is the outward normal derivation on $\partial\Omega$.

The variational problem associated with (2.1) is given by: Find $\lambda \in \mathbb{R}$ and $0 \neq u \in H^1(\Omega)$, such that

$$a(u,v) = \lambda b(u,v), \forall v \in H^1(\Omega) \tag{2.2}$$

where

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)dx, b(u,v) = \int_{\partial\Omega} uvds$$

$$\| u \|_b = (b(u,u))^{\frac{1}{2}} = \| u \|_{0,\partial\Omega}$$

It is clear that $a(\cdot,\cdot)$ is symmetric, continuous and $H^1(\Omega)$- elliptic bilinear form on $H^1(\Omega) \times H^1(\Omega)$.

Let $\mathcal{T}_h = \{T\}$ be a family of regular triangulations of Ω. Let h stand for the mesh-size, namely $h = max\{h_T: T \in \mathcal{T}_h\}$ is the diameter of $\mathcal{T}_h$, with h_T being the diameter of the triangle T. The diameter of an edge e is denoted by h_e, and the set of edges of elements $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$ where $\mathcal{E}_h^i$ denotes the interior edges set and $\mathcal{E}_h^b$ denotes the set of edges lying on the boundary $\partial\Omega$. We denote the average $\{v\}$ and jump $[[v]]$ of v on e by

$$\{v\} = \frac{1}{2}(v^+ + v^-), \qquad [[v]] = v^+\mathbf{n}^+ + v^-\mathbf{n}^-$$

where $e \in \partial T^+ \cap \partial T^-, v^+ = v|_{T^+}, v^- = v|_{T^-}$, $\mathbf{n}$ is the unit outer normal vector from T^+ towards to T^-.

If $e \in \mathcal{E}_h^b$, define the average and jump of v on e as follows:

$$\{v\} = v, \qquad [[v]] = v\mathbf{n}$$

Define the DGFEM space:

$$S^h = \{v \in L^2(\Omega): v|_T \in \mathbb{P}_m(T), \forall T \in \mathcal{T}_h\}$$

where $\mathbb{P}_m(T)$ denotes the space of polynomials defined on T with degree less than or equal to $m \geq 1$. Introduce the piecewise H^s function space of degree s:

$$H^s(\mathcal{T}_h) = \{v \in L^2(\Omega): v\,|_T \in H^s(T), \forall T \in \mathcal{T}_h\}$$

The DGFEM discretization of (2.2) is to find $\lambda_h \in \mathbb{R}$ and $0 \neq u_h \in S^h$, such that

$$a_h(u_h, v_h) = \lambda_h b_h(u_h, v_h), \forall v_h \in S^h \tag{2.3}$$

where

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} \int_T (\nabla u_h \cdot \nabla v_h + u_h v_h)\mathrm{d}x - \sum_{e \in \mathcal{E}_h^i} \int_e \{\nabla u_h\} \cdot [[v_h]]\mathrm{d}s$$

$$- \sum_{e \in \mathcal{E}_h^i} \int_e \{\nabla v_h\} \cdot [[u_h]]\mathrm{d}s + \sum_{e \in \mathcal{E}_h^i} \frac{\sigma}{h_e} \int_e [[u_h]] \cdot [[v_h]]ds$$

$$b_h(u_h, v_h) = \sum_{e \in \mathcal{E}_h^b} \int_e u_h \cdot v_h \mathrm{d}s$$

where σ is the interior penalty parameter. We choose σ to be sufficiently large to have coercivity. It is clear that the discretization (2.3) is symmetric which is called symmetric interior penalty Galerkin method (SIPG) in DGFEM.

Introduce the sum space $V(h) = S^h + H^1(\Omega)$ endowed with DG norm

$$\|u_h\|_G^2 = \sum_{T \in \mathcal{T}_h} \left(\|\nabla u_h\|_{0,T}^2 + \|u_h\|_{0,T}^2\right) + \sum_{e \in \mathcal{E}_h^i} \frac{\sigma}{h_e} \|[[u_h]]\|_{0,e}^2$$

and define the other norm on $H^{1+s}(\mathcal{T}_h)$ $\left(s > \frac{1}{2}\right)$ by

$$\|u_h\|_h^2 = \|u_h\|_G^2 + \sum_{e \in \mathcal{E}_h^i} h_e \|\{\nabla u_h\}\|_{0,e}^2$$

Note that $\|\cdot\|_G$ is equivalent to $\|\cdot\|_h$ on S^h.

In order to show that the discretization (2.3) is stable, first we will show that $a_h(\cdot,\cdot)$ is coercive on $S^h \times S^h$. It is easy to know that the following continuity and coercivity properties hold:

$$|a_h(u_h, v_h)| \lesssim \|u_h\|_h \|v_h\|_h, \forall u_h, v_h \in S^h + H^{1+s}(\mathcal{T}_h)\left(s > \frac{1}{2}\right) \tag{2.4}$$

$$\| u_h \|_G^2 \lesssim |a_h(u_h, u_h)|, \forall u_h \in S^h \tag{2.5}$$

Proof. Combining the Cauchy-Schwarz inequality, we obtain inequality (2.4). For convenience, for any $u_h \in S^h$, we set

$$\begin{aligned} a_h(u_h, u_h) &= \sum_{T\in\mathcal{T}_h} \| \nabla u_h \|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \| u_h \|_{0,T}^2 \\ &\quad -2\sum_{e\in\mathcal{E}_h^i} \int_e \{\nabla u_h\} \cdot [[u_h]] ds + \sum_{e\in\mathcal{E}_h^i} \sigma \| h_e^{-\frac{1}{2}} [[u_h]] \|_{0,e}^2 \\ &= I + II + III + IV \end{aligned} \tag{2.6}$$

To show the coercivity we need to bound the potentially negative terms by the positive terms. We first estimate the third term. Using Cauchy-Schwarz inequality, the inverse inequality and Young's inequality, we deduce that

$$\begin{aligned} \sum_{e\in\mathcal{E}_h^i} \int_e \{\nabla u_h\} \cdot [[u_h]] ds &= \sum_{T\in\mathcal{T}_h} \int_{\partial T\backslash\partial\Omega} \{\nabla u_h\} \cdot [[u_h]] ds \\ &\leq \sum_{T\in\mathcal{T}_h} \| h_e^{\frac{1}{2}} \{\nabla u_h\} \|_{0,\partial T\backslash\partial\Omega} \| h_e^{-\frac{1}{2}} [[u_h]] \|_{0,\partial T\backslash\partial\Omega} \\ &\lesssim \sum_{T\in\mathcal{T}_h} \| h_e^{\frac{1}{2}} \nabla u_h \|_{0,\partial T\backslash\partial\Omega} \| h_e^{-\frac{1}{2}} [[u_h]] \|_{0,\partial T\backslash\partial\Omega} \\ &\lesssim \sum_{T\in\mathcal{T}_h} \sqrt{C} \| \nabla u_h \|_{0,T} \| h_e^{-\frac{1}{2}} [[u_h]] \|_{0,\partial T\backslash\partial\Omega} \\ &\lesssim \sum_{T\in\mathcal{T}_h} \left(\delta C \| \nabla u_h \|_{0,T}^2 + \frac{1}{4\delta} \| h_e^{-\frac{1}{2}} [[u_h]] \|_{0,\partial T\backslash\partial\Omega}^2\right) \end{aligned}$$

and taking $\delta = (1-\beta)/2C$ and $\sigma \geq \beta\sigma + 1/(2\delta) = C/(1-\beta)^2$ we deduce

$$\begin{aligned} a_h(u_h, u_h) &= \sum_{T\in\mathcal{T}_h} \| \nabla u_h \|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \| u_h \|_{0,T}^2 \\ &\quad -2\sum_{e\in\mathcal{E}_h^i} \int_e \{\nabla u_h\}[[u_h]] ds + \sum_{e\in\mathcal{E}_h^i} \sigma \| h_e^{-\frac{1}{2}} [[u_h]] \|_{0,e}^2 \\ &\geq \sum_{T\in\mathcal{T}_h} (\| \nabla u_h \|_{0,T}^2 + \| u_h \|_{0,T}^2 - 2\delta C \| \nabla u_h \|_{0,T}^2 \\ &\quad -\frac{1}{2\delta} \| h_e^{-\frac{1}{2}} [[u_h]] \|_{0,\partial T\backslash\partial\Omega}^2 + \sigma \| h_e^{-\frac{1}{2}} [[u_h]] \|_{0,\partial T\backslash\partial\Omega}^2) \\ &= \sum_{T\in\mathcal{T}_h} \left((1-2\delta C) \| \nabla u_h \|_{0,T}^2 + (\sigma - \frac{1}{2\delta}) \| h_e^{-\frac{1}{2}} [[u_h]] \|_{0,\partial T\backslash\partial\Omega}^2 + \| u_h \|_{0,T}^2\right) \\ &\geq \beta \sum_{T\in\mathcal{T}_h} \left(\| \nabla u_h \|_{0,T}^2 + \| u_h \|_{0,T}^2 + \sigma \| h_e^{-\frac{1}{2}} [[u_h]] \|_{0,\partial T\backslash\partial\Omega}^2\right) \\ &= \beta \| u_h \|_G^2 \end{aligned}$$

So the coerciveness of $a_h(\cdot,\cdot)$ is valid.

We consider the following source problem (2.7) associated with (2.2) and the DG approximate source problem (2.8) associated with (2.3), respectively.

Find $w \in H^1(\Omega)$ such that

$$a(w, v) = b(f, v), \forall v \in H^1(\Omega) \tag{2.7}$$

Find $w_h \in S^h$ such that

$$a_h(w_h, v_h) = b_h(f, v_h), \forall v_h \in S^h \tag{2.8}$$

Since $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ are continuous and coercive on $H^1(\Omega)$ and S^h, respectively. $b(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ are bounded, from Lax-Milgram Theorem we know that (2.7) and (2.8) admit the unique solution w and w_h, respectively.

We now recall the following regularity estimates for the above source problem [31].

Lemma 2.1. If $f\in L^2(\partial\Omega)$, the solution w of the source problem (2.7) satisfies $w\in H^{1+r_1}(\Omega)$ with $r_1\in(0,\frac{1}{2})$ and

$$\| w \|_{1+r_1}\lesssim\| f \|_{0,\partial\Omega} \tag{2.9}$$

For the case that $f\in H^{\frac{1}{2}}(\partial\Omega)$, we have $w\in H^{1+r_2}(\Omega)$ and

$$\| w \|_{1+r_2}\lesssim\| f \|_{\frac{1}{2},\partial\Omega} \tag{2.10}$$

Let w and w_h be the solution of (2.7) and (2.8), respectively, then the SIPG approximation (2.8) is consistent:

$$a_h(w-w_h,v_h)=0,\forall v_h\in S^h \tag{2.11}$$

Proof. Applying Green's formula elementwise in $\mathcal{T}_h$, and using the fact that $\sum_{T\in\mathcal{T}_h}\int_{\partial T} q_K\varphi_K\cdot n_K ds=\int_{\mathcal{E}_h}[[q]]\cdot\{\varphi\}ds+\int_{\mathcal{E}_h^i}\{q\}\cdot[[\varphi]]ds$ and $\int_e[[\nabla w]]\cdot v=0$ on inner edge e, we deduce

$$\begin{aligned}
0&=\sum_{T\in\mathcal{T}_h}\int_T(-\Delta w+w)vdx\\
&=\sum_{T\in\mathcal{T}_h}\int_T\nabla w\cdot\nabla vdx+\sum_{T\in\mathcal{T}_h}\int_T wvdx-\sum_{T\in\mathcal{T}_h}\int_{\partial T}\frac{\partial w}{\partial\mathbf{n}}\cdot vds\\
&=\sum_{T\in\mathcal{T}_h}\int_T(\nabla w\cdot\nabla v+wv)dx-\sum_{T\in\mathcal{T}_h}\int_{\partial T}\nabla w\cdot\mathbf{n}\cdot vds\\
&=\sum_{T\in\mathcal{T}_h}\int_T(\nabla w\cdot\nabla v+wv)dx-\sum_{e\in\mathcal{E}_h^i}\int_e\{\nabla w\}\cdot[[v]]ds-\sum_{e\in\mathcal{E}_h}\int_e[[\nabla w]]\cdot\{v\}ds\\
&=\sum_{T\in\mathcal{T}_h}\int_T(\nabla w\cdot\nabla v+wv)dx-\sum_{e\in\mathcal{E}_h^i}\int_e\{\nabla w\}\cdot[[v]]ds-\sum_{e\in\mathcal{E}_h^b}\int_e[[\nabla w]]\cdot vds\\
&=\sum_{T\in\mathcal{T}_h}\int_T(\nabla w\cdot\nabla v+wv)dx-\sum_{e\in\mathcal{E}_h^i}\int_e\{\nabla w\}\cdot[[v]]ds-\sum_{e\in\mathcal{E}_h^i}\int_e\{\nabla v\}\cdot[[w]]ds\\
&\quad+\sum_{e\in\mathcal{E}_h^i}\frac{\sigma}{h_e}\int_e[[w]]\cdot[[v]]ds-\sum_{e\in\mathcal{E}_h^b}\int_e fvds
\end{aligned}$$

which means that

$$a_h(w,v)=b_h(f,v),\forall v\in H^{1+r}(\mathcal{T}_h) \tag{2.12}$$

It is obvious that $S^h\subset H^{1+r}(\mathcal{T}_h)$, then subtracting (2.8) from (2.12) we get (2.11).

Then, thanks to Lemma 2.1, for the source problem (2.7), let $f\in L^2(\partial\Omega)$, we can define the solution

operator $A: L^2(\partial\Omega)\to H^{1+\frac{r}{2}}(\Omega)\subset H^1(\Omega)$ as

Then, thanks to Lemma 2.1, for the source problem (2.7), let $f\in L^2(\partial\Omega)$, we can define the solution operator $A: L^2(\partial\Omega)\to H^{1+\frac{r}{2}}(\Omega)\subset H^1(\Omega)$ as

$$a(Af,v)=b(f,v),\forall v\in H^1(\Omega) \tag{2.13}$$

Define the operator $T: L^2(\partial\Omega)\to H^{\frac{1}{2}+\frac{r}{2}}(\partial\Omega)$, such that

$$Tf=Af\mid_{\partial\Omega}$$

Similarly, from (2.8) we define a discrete solution operator $A_h: L^2(\partial\Omega)\to S^h$ as

$$a_h(A_hf,v)=b_h(f,v),\forall v\in S^h \tag{2.14}$$

and the discrete operator $T_h: L^2(\partial\Omega)\to\delta S^h\subset L^2(\partial\Omega)$, such that

$$T_hf=A_hf|_{\partial\Omega}$$

where δS^h is the restriction of S^h on $\partial\Omega$.

Hence, (2.2) and (2.3) has the following equivalent operator form, respectively:

$$Au=\mu u,Tu=\mu u \tag{2.15}$$

$$A_hu_h=\mu_hu_h,T_hu_h=\mu_hu_h \tag{2.16}$$

where $\mu=\frac{1}{\lambda},\mu_h=\frac{1}{\lambda_h}$. In this paper, λ, λ_h and μ, μ_h are all called eigenvalues.

From the definition of A_h and (2.5), noticing that $\|\cdot\|_G$ is equivalent to $\|\cdot\|_h$ on S^h, we can derive that

$$\| A_hf \|_h^2\lesssim a_h(A_hf,A_hf)=b_h(f,A_hf)\lesssim\| f \|_{0,\partial\Omega}\| A_hf \|_{0,\partial\Omega}\lesssim\| f \|_{0,\partial\Omega}\| A_hf \|_h$$

which yields

$$\| A_hf \|_h\lesssim\| f \|_{0,\partial\Omega}\lesssim\| f \|_h \tag{2.17}$$

Lemma 2.2. Suppose that $\varphi \in H^{1+\xi}(T)(0 < \xi < \frac{1}{2})$ and $\Delta\varphi \in L^2(T)$, then there holds

$$\| \nabla\varphi \cdot \mathbf{n} \|_{\xi-\frac{1}{2},e} \lesssim \| \nabla\varphi \|_{\xi,T} + h_T^{1-\xi} \| \Delta\varphi \|_{0,T}, \forall T \in \mathcal{T}_h, e \in \partial T \tag{2.18}$$

Introduce the auxiliary problem: find $\psi \in H^1(\Omega)$ such that

$$a_h(v,\psi) = (v,g), \forall v \in H^1(\Omega) \tag{2.19}$$

From the elliptic regularity estimates for homogeneous Neumann boundary problem, we know that the following regularity estimate holds: $\forall g \in L^2(\Omega)$, the solution ψ of (2.18) belongs to $H^{1+\beta}(\Omega)(\beta > \frac{1}{2})$ and satisfies

$$\| \psi \|_{1+\beta} \lesssim \| g \|_{0,\Omega} \tag{2.20}$$

Let $\psi^I \in S^h$ denote the linear interpolation of ψ on $\mathcal{T}_h$.

Lemma 2.3. Suppose that w and w_h be the solution of (2.7) and (2.8), respectively, $w \in H^{1+s}(\Omega)(0 < s < \frac{1}{2})$, then there hold

$$\| w - w_h \|_{0,\Omega} \lesssim h^\beta \| w - w_h \|_G \tag{2.21}$$

Proof. For any fixed $g \in L^2(\Omega)$, by using the consistency of DG method, (2.11) and the Schwarz inequality we deduce

$$\begin{aligned}
(g, w - w_h) &= a_h(w - w_h, \psi) = a_h(w - w_h, \psi - \psi^I) \\
&\lesssim \| w - w_h \|_G \| \psi - \psi^I \|_G + \Big| \sum_e \int_e [[w - w_h]] \cdot \{\nabla(\psi - \psi^I)\} ds \Big| \\
&\lesssim h^\beta \| w - w_h \|_G \| \psi \|_{1+\beta} + \Big| \sum_e \int_e [[w - w_h]] \cdot \{\nabla(\psi - \psi^I)\} ds \Big| \\
&\lesssim h^\beta \| w - w_h \|_G \| \psi \|_{1+\beta} + \sum_e \| [[w - w_h]] \|_{0,e} \| \{\nabla(\psi - \psi^I)\} \|_{0,e}
\end{aligned} \tag{2.22}$$

From the trace inequality, the interpolation estimate, the definition of DG norm and (2.20), we get

$$\begin{aligned}
\sum_e \| [[w - w_h]] \|_{0,e} \| \{\nabla(\psi - \psi^I)\} \|_{0,e} &\lesssim \sum_e \| [[w - w_h]] \|_{0,e}\, h_e^{\beta-\frac{1}{2}} \| \psi \|_{1+\beta,T^+\cup T^-} \\
&\lesssim \left(\sum_e \| h_e^{-\frac{1}{2}} [[w - w_h]] \|_{0,e}^2 \right)^{\frac{1}{2}} h^\beta \| \psi \|_{1+\beta} \\
&\lesssim h^\beta \| w - w_h \|_G \| g \|_{0,\Omega}
\end{aligned} \tag{2.23}$$

Substituting (2.23) into (2.22) and using the Riesz representation theorem, we get (2.21). The proof is completed.

Theorem 2.1. Suppose that w and w_h be the solution of (2.7) and (2.8), respectively, $w \in H^{1+s}(\Omega)(0 < s < \frac{1}{2})$, then there hold

$$\| w - w_h \|_G \lesssim h^s \| w \|_{1+s,\Omega} \tag{2.24}$$

Proof. From (2.5), (2.11), the definition of $a_h(\cdot,\cdot)$ and the Schwarz inequality, it is obtained that

$$\begin{aligned}
\|w^I - w_h\|_G^2 &\lesssim |a_h(w^I - w, w^I - w_h)| \\
&\lesssim \|w^I - w\|_G \|w^I - w_h\|_G + \Big| \sum_e \int_e \{\nabla(w^I - w)\} \cdot \big[[w^I - w_h]\big] \mathrm{d}s \Big| \\
&\lesssim h^s \|w^I - w_h\|_G \|w\|_{1+s} + \sum_e \|\{\nabla(w^I - w)\}\|_{s-\frac{1}{2},e} \big\|\big[[w^I - w_h]\big]\big\|_{\frac{1}{2}-s,e}.
\end{aligned} \tag{2.25}$$

Derived from inverse estimate, trace estimate, interpolation estimate and (2.21), we deduce

$$\begin{aligned}
\sum_e \|\{\nabla(w^I - w)\}\|_{s-\frac{1}{2},e} \|[[w^I - w_h]]\|_{\frac{1}{2}-s,e} &\lesssim \sum_e h_e^{s-\frac{1}{2}} \| [[w^I - w_h]] \|_{0,e} \|\{\nabla(w^I - w)\}\|_{s-\frac{1}{2},e} \\
&\lesssim \sum_e h_e^{s-\frac{1}{2}} \| [[w^I - w_h]] \|_{0,e} \| w^I - w \|_{s+\frac{1}{2},e} \\
&\lesssim \sum_e h_e^{s-\frac{1}{2}} \| \big[[w^I - w_h]\big] \|_{0,e} \| w \|_{1+s}
\end{aligned} \tag{2.26}$$

$$\lesssim \left(\sum_e \| h_e^{-\frac{1}{2}}[[w^I - w_h]] \|_{0,e}^2\right)^{\frac{1}{2}} h^s \| w \|_{1+s}$$
$$\lesssim h^s \| w^I - w_h \|_G \| w \|_{1+s}$$

then using (2.25), (2.26), triangle inequality and the interpolation error estimate we obtain (2.24), the proof is completed.

Let $w^{*I} \in S^h$ denote the linear interpolation of w^* on $\mathcal{T}_h$.

Theorem 2.2. Suppose that w and w_h be the solution of (2.7) and (2.8), respectively, $w \in H^{1+s}(\Omega)(0 < s < \frac{1}{2})$, then there hold

$$\| w - w_h \|_{0,\partial\Omega} \lesssim h^s \| w - w_h \|_G \tag{2.27}$$

Proof. Consider the source problem of the dual problem of equation (2.2) $a(v, w^*) = \langle v, g\rangle, \forall v \in H^1(\Omega)$, where $\langle v, g\rangle = \int_{\partial\Omega} vgds$, for any fixed $g \in L^2(\partial\Omega)$, using (2.11) we obtain

$$\begin{aligned}\langle w - w_h, g\rangle &= a_h(w - w_h, w^*) = a_h(w - w_h, w^* - w^{*I}) \\ &\lesssim h^s \|w - w_h\|_G \|w^*\|_{1+s} + |\sum_e \int_e [[w - w_h]] \cdot \{\nabla(w^* - w^{*I})\}\mathrm{d}s|\end{aligned} \tag{2.28}$$

Then

$$\begin{aligned}|\sum_e \int_e [[w - w_h]] \cdot \{\nabla(w^* - w^{*I})\}ds| &\lesssim \sum_e \| [[w - w_h]] \|_{\frac{1}{2}-s,e} \| \{\nabla(w^* - w^{*I})\} \|_{s-\frac{1}{2},e} \\ &\lesssim \sum_e h_e^{s-\frac{1}{2}} \| [[w - w_h]] \|_{0,e} \|\{\nabla(w^* - w^{*I})\}\|_{s-\frac{1}{2},e} \\ &\lesssim \sum_e h_e^{s-\frac{1}{2}} \| [[w - w_h]] \|_{0,e} \| w^* - w^{*I} \|_{s+\frac{1}{2},e} \\ &\lesssim \sum_e h_e^{s-\frac{1}{2}} \| [[w - w_h]] \|_{0,e} \| w^* \|_{1+s} \\ &\lesssim h^s \| w - w_h \|_G \| g \|_{0,\partial\Omega}\end{aligned} \tag{2.29}$$

Substituting (2.29) into (2.28), and using the Riesz representation theorem we get (2.27).

Assume that λ is the kth eigenvalue of (2.2), and the algebraic multiplicity is equal to q, $\lambda = \lambda_k = \lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_{k+q-1}$. When $\| T - T_h \|_{0,\partial\Omega} \to 0 (h \to 0)$ [5], q eigenvalues $\lambda_{k,h}$, $\lambda_{k+1,h}$, $\cdots$, $\lambda_{k+q-1,h}$ of (2.3) will converge to λ. Let $M(\lambda)$ be the space spanned by all eigenfunctions corresponding to λ and $M_h(\lambda)$ be the direct sum of the eigenspaces corresponding to all eigenvalues of (2.3) that converge to λ. We have the following error estimates [32].

Theorem 2.3. We assume that $M(\lambda) \subset H^{s+1}(\Omega)\left(s > \frac{1}{2}\right)$, $t = min(m, s)$, then there holds

$$|\lambda - \lambda_h| \lesssim h^{2t} \tag{2.30}$$

Let $u_h \in M_h(\lambda)$ be an eigenfunction of (2.3), then there exists $u \in M(\lambda)$ such that

$$\| u - u_h \|_{0,\partial\Omega} \lesssim h^{t+r} \tag{2.31}$$

$$\| u - u_h \|_h \lesssim h^t \tag{2.32}$$

Proof. Let λ and λ_h be the kth eigenvalue of (2.2) and (2.3), respectively, and $\dim M(\lambda) = q$. From Theorem 7.3 [15] we have

$$|\lambda - \lambda_h| \lesssim \sum_{i,j=k}^{k+q-1} |b_h\left((T - T_h)\varphi_i, \varphi_j\right)| + \| (T - T_h)|_{M(\lambda)} \|_{0,\partial\Omega}^2 \tag{2.33}$$

where $\varphi_k, \cdots, \varphi_{k+q-1}$ are the basis functions for $M(\lambda)$. Then, from (2.12), (2.11) and (2.4), we deduce

$$\begin{aligned}b_h((T - T_h)\varphi_i, \varphi_j) &= b_h((A - A_h)\varphi_i, \varphi_j) = a_h(A\varphi_i - A_h\varphi_i, A\varphi_j) \\ &= a_h(A\varphi_i - A_h\varphi_i, A\varphi_j - A_h\varphi_j) \\ &\lesssim \| A\varphi_i - A_h\varphi_i \|_h \| A\varphi_j - A_h\varphi_j \|_h \\ &\lesssim h^t \| A\varphi_i \|_{1+t} h^t \| A\varphi_j \|_{1+t} \lesssim h^{2t}\end{aligned} \tag{2.34}$$

Noting that $Af = w, A_h f = w_h$, using Theorem 2.2 and Theorem 2.1, we get

$$\begin{aligned}\| (T - T_h)|_{M(\lambda)} \|_{0,\partial\Omega} &= \sup_{f \in M(\rho), \|f\|_{0,\partial\Omega}=1} \| Tf - T_h f \|_{0,\partial\Omega} \\ &\lesssim \sup_{f \in M(\rho), \|f\|_{0,\partial\Omega}=1} h^{t+r} \| Af \|_{1+t,\Omega} \lesssim h^{t+r}\end{aligned} \tag{2.35}$$

Substituting (2.34) and (2.35) into (2.33), we obtain (2.30).

Since $\| T - T_h \|_{0,\partial\Omega} \to 0$, from the spectral approximation theory we know that there exists $u \in M(\lambda)$ such that

$$\| u - u_h \|_{0,\partial\Omega} \leq C \| (T - T_h) |_{M(\lambda)} \|_{0,\partial\Omega} \quad (2.36)$$

Then (2.31) follows directly from (2.36) and (2.35).

Since $u = \lambda A u$ and $u_h = \lambda_h A_h u_h$, using the triangular inequality, (2.17), (2.30) and (2.31), we deduce

$$\begin{aligned} \| u_h - u \|_h &= \| \lambda_h A_h u_h - \lambda A u \|_h \\ &\lesssim \| \lambda_h A_h u_h - \lambda A_h u \|_h + \| \lambda A_h u - \lambda A u \|_h \\ &\lesssim \| \lambda_h u_h - \lambda u \|_{0,\partial\Omega} + \| A_h u - A u \|_h \\ &\lesssim |\lambda_h - \lambda| + \| u_h - u \|_{0,\partial\Omega} + \| A_h u - A u \|_h \\ &\lesssim h^{2t} + h^{t+r} + h^t \lesssim h^t \end{aligned}$$

i.e., (2.32) is valid. The proof is completed.

III. TWO-GRID DISCRETIZATION

Let $\{\mathcal{T}_{h_i}\}_0^l$ be an family of regular meshes of Ω, $h_{i-1} \gg h_i$, and let S^{h_i} be the DG space defined on $\mathcal{T}_{h_i}$. Denote $\mathcal{T}_{h_0} = \mathcal{T}_H$, $S^{h_0} = S^H$. Now, for the eigenvalue problem (2.3) we give the following two-grid discretization scheme of DGFEM based on the shifted inverse iteration.

Scheme 3.1. Given the iterative times l.

Step 1: Solve (2.3) on S^H: Find $(\lambda_H, u_H) \in \mathbb{R} \times S^H$ such that $\| u_H \|_{0,\partial\Omega} = 1$ and

$$a_H(u_H, v) = \lambda_H b_H(u_H, v), \forall v \in S^H \quad (3.1)$$

Step 2: $u^{h_0} \Leftarrow u_H, \lambda^{h_0} \Leftarrow \lambda_H, i \Leftarrow 1$.

Step 3: Solve a linear system on S^{h_i}: Find $u' \in S^{h_i}$ such that

$$a_{h_i}(u', v) - \lambda^{h_{i-1}} b_{h_i}(u', v) = b_{h_i}(u^{h_{i-1}}, v), \forall v \in S^{h_i} \quad (3.2)$$

Let $u^{h_i} = \frac{u'}{\|u'\|_{0,\partial\Omega}}$.

Step 4: Compute the Rayleigh quotient

$$\lambda^{h_i} = \frac{a_{h_i}(u^{h_i}, u^{h_i})}{b_{h_i}(u^{h_i}, u^{h_i})}$$

Step 5: If $i = l$, then output (λ^{h_l}, u^{h_l}), stop; else, $i \Leftarrow i + 1$ and return to Step 3.

From (2.11) we define the Ritz-Galerkin projection operator $P_h: H^1(\Omega) \to S^h$ by

$$a_h(u - P_h u, v_h) = 0, \forall v_h \in S^h \quad (3.3)$$

Hence, for any $f \in H^1(\Omega)$

$$a_h(A_h f - P_h(Af), v_h) = a_h(A_h f - Af + Af - P_h(Af), v_h) = 0, \forall v_h \in S^h$$

Then, $A_h f = P_h A f, \forall v \in H^1(\Omega)$, thus $A_h = P_h A$.

We first give the following lemmas to prepare for the error analysis.

Lemma 3.1. Let (λ, u) be an eigenpair of (2.2), then for any $v \in S^h$ and $\| v \|_b \neq 0$, the Rayleigh quotient $R(v) = \frac{a_h(v,v)}{\|v\|_b^2}$ such that

$$\frac{a_h(v, v)}{\| v \|_b^2} - \lambda = \frac{a_h(v - u, v - u)}{\| v \|_b^2} - \lambda \frac{\| v - u \|_b^2}{\| v \|_b^2} \quad (3.4)$$

Proof. From (2.11), we deduce

$$a_h(u, v) = b(\lambda u, v) = b_h(\lambda u, v), \forall v \in S^h$$

therefore,

$$\begin{aligned} &a_h(v - u, v - u) - \lambda b(v - u, v - u) \\ &= a_h(v, v) + a_h(u, u) - 2a_h(v, u) - \lambda b(v, v) - \lambda b(u, u) + 2\lambda b(v, u) \\ &= a_h(v, v) + \lambda b(u, u) - 2\lambda b(v, u) - \lambda b(v, v) - \lambda b(u, u) + 2\lambda b(v, u) \\ &= a_h(v, v) - \lambda b(v, v) \end{aligned}$$

By dividing by $\| v \|_b^2$ on both sides of the above identity, we have (3.4).

Lemma 3.2. For any nonzero $u, v \in S^h$,

$$\left\| \frac{u}{\| u \|} - \frac{v}{\| v \|} \right\| \leq 2 \frac{\| u - v \|}{\| u \|}, \left\| \frac{u}{\| u \|} - \frac{v}{\| v \|} \right\| \leq 2 \frac{\| u - v \|}{\| v \|} \quad (3.5)$$

Proof.

$$\begin{aligned} \left\| \frac{u}{\| u \|} - \frac{v}{\| v \|} \right\| &= \left\| \frac{u \| v \| - v \| u \|}{\| u \| \| v \|} \right\| = \left\| \frac{u \| v \| - v \| v \| + v \| v \| - v \| u \|}{\| u \| \| v \|} \right\| \\ &\leq \frac{\| (u - v) \| v \| + v(\| v \| - \| u \|) \|}{\| u \| \| v \|} \leq 2 \frac{\| u - v \|}{\| u \|} \end{aligned}$$

i.e., the first inequality holds. Similarly, one can deduce the second inequality. Note that Lemma 3.2 holds in any normed space.

Let (λ_H, u_H) be the kth eigenpair of (3.1), then (λ^{h_l}, u^{h_l}) derived from Scheme 3.1 is the kth eigenpair approximation of (2.2). In what follows we also denote $(\lambda_H, u_H) = (\lambda_{k,H}, u_{k,H})$, $(\lambda^{h_l}, u^{h_l}) = (\lambda_k^{h_l}, u_k^{h_l})$.

IV. THE THEORETICAL ANALYSIS

Next, we will prove the error estimates and the convergence of (λ^{h_l}, u^{h_l}) derived from Scheme 3.1. Our analysis is based on the following crucial property of the shifted-inverse iteration in DGFEM.

In the following discussion, let (λ_k, u_k) and $(\lambda_{k,h}, u_{k,h})$ denote the kth eigenpair of (2.2) and (2.3), respectively, and $\mu_k = \frac{1}{\lambda_k}$, $\mu_{k,h} = \frac{1}{\lambda_{k,h}}$, $M(\mu_k) = M(\lambda_k)$, $M_h(\mu_k) = M_h(\lambda_k)$.

Denote $dist(u, S^h) = \inf_{v \in S^h} \| u - v \|_h$, $d = dimS^h$.

Lemma 4.1. Let (μ_0, w_0) be an approximation of the kth eigenpair (μ, u) of (2.2), where μ_0 is not an eigenvalue of A_h, and $w_0 \in S^h$ with $\| w_0 \|_{0,\partial\Omega} = 1$. And let $u_0 = \frac{A_h w_0}{\|A_h w_0\|_{0,\partial\Omega}}$. Suppose that

$(C1)$ $\inf_{v \in M_h(\lambda)} \| w_0 - v \|_{0,\partial\Omega} \le \frac{1}{2}$;

$(C2)$ $|\mu_0 - \mu| \le \frac{\rho}{4}$, $|\mu_{j,h} - \mu_j| \le \frac{\rho}{4}$ for $j = k-1, k, k+q (j \ne 0)$, $\rho = \min_{j \ne k} |\mu_j - \mu|$ is the separate constant of the eigenvalue μ.

$(C3)$ $u' \in S^h$ and $u^h \in S^h$ satisfy

$$(\mu_0 - A_h)u' = u_0, u^h = \frac{u'}{\| u' \|_{0,\partial\Omega}} \tag{4.1}$$

Then

$$dist\big(u^h, M_h(\lambda)\big) \le \frac{C}{\rho} \max_{k \le j \le k+q-1} \mu_0 - \mu_{j,h} \; dist\big(w_0, M_h(\lambda)\big) \tag{4.2}$$

Proof. Let $\{u_{j,h}\}_1^d$ be eigenfunctions of A_h satisfying $b(u_{j,h}, u_{i,h}) = \delta_{i,j}$. Then

$$u_0 = \sum_{j=1}^{d} b(u_0, u_{j,h}) u_{j,h}$$

Since μ_0 is not an eigenvalue of A_h, from (4.1) we can obtain

$$\big(\mu_0 - \mu_{k,h}\big)u' = \big(\mu_0 - \mu_{k,h}\big)(\mu_0 - T_h)^{-1} u_0 = \sum_{j=1}^{d} \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} b\big(u_0, u_{j,h}\big) u_{j,h} \tag{4.3}$$

Using the triangle inequalities and the condition $(C2)$, we have

$$|\mu_0 - \mu_{k,h}| \le |\mu_0 - \mu| + |\mu - \mu_{k,h}| \le \frac{\rho}{4} + \frac{\rho}{4} = \frac{\rho}{2}$$

$$|\mu_0 - \mu_{j,h}| \ge |\mu - \mu_j| - |\mu_0 - \mu| - |\mu_j - \mu_{j,h}| \ge \rho - \frac{\rho}{4} - \frac{\rho}{4} = \frac{\rho}{2}$$

where $j = k-1, k+q (j \ne 0)$, and thus we obtain

$$\mu_0 - \mu_{j,h} \ \ge \frac{\rho}{2} \ \ for \ j \ne k, k+1, \cdots, k+q-1 \tag{4.4}$$

Because the operator T_h is self-adjoint with respect to $b(\cdot,\cdot)$, in fact, for $\forall f \in L^2(\partial\Omega)$, from the symmetry of $a_h(\cdot,\cdot)$ and $b(\cdot,\cdot) = b_h(\cdot,\cdot)$ we have

$$\begin{aligned} b(T_h f, v_h) &= b(v_h, T_h f) = b_h(v_h, T_h f) = a_h(T_h v_h, T_h f) \\ &= a_h(T_h f, T_h v_h) = b_h(f, T_h v_h) = b(f, T_h v_h) \end{aligned}$$

and $A_h u_h = \mu_h u_h$, therefore, for $j = 1, 2, \cdots, d$, there holds

$$\begin{aligned} b(T_h w_0, u_{j,h}) u_{j,h} &= b(w_0, T_h u_{j,h}) u_{j,h} = b(w_0, \mu_{j,h} u_{j,h}) u_{j,h} \\ &= b(w_0, u_{j,h}) \mu_{j,h} u_{j,h} = b(w_0, u_{j,h}) A_h u_{j,h} \end{aligned} \tag{4.5}$$

Noticing that $\{u_{j,h}\}_k^{k+q-1}$ is an orthonormal basis of $M_h(\lambda)$ with respect to the $L^2(\partial\Omega)$ inner product $b(\cdot,\cdot)$, from $u_0 = \frac{A_h w_0}{\|A_h w_0\|_{0,\partial\Omega}}$, (4.3), (4.5), (2.17) and (4.4) we derive

$$\begin{aligned} &\| \big(\mu_0 - \mu_{k,h}\big)u' - \sum_{j=k}^{k+q-1} \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} b\big(u_0, u_{j,h}\big) u_{j,h} \|_h \\ &= \| \sum_{j \ne k, k+1, \cdots, k+q-1} \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} b\big(u_0, u_{j,h}\big) u_{j,h} \|_h \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\| A_h w_0 \|_{0,\partial\Omega}} \left\| \sum_{j \neq k,k+1,\cdots,k+q-1} \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} b(A_h w_0, u_{j,h}) u_{j,h} \right\|_h \\
&= \frac{1}{\| A_h w_0 \|_{0,\partial\Omega}} \left\| \sum_{j \neq k,k+1,\cdots,k+q-1} \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} b(T_h w_0, u_{j,h}) u_{j,h} \right\|_h \\
&= \frac{1}{\| A_h w_0 \|_{0,\partial\Omega}} \left\| \sum_{j \neq k,k+1,\cdots,k+q-1} \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} b(w_0, u_{j,h}) A_h u_{j,h} \right\|_h \\
&= \frac{1}{\| A_h w_0 \|_{0,\partial\Omega}} \left\| A_h \left(\sum_{j \neq k,k+1,\cdots,k+q-1} \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} b(w_0, u_{j,h}) u_{j,h} \right) \right\|_h \qquad (4.6) \\
&\leq \frac{C}{\| A_h w_0 \|_{0,\partial\Omega}} \left\| \sum_{j \neq k,k+1,\cdots,k+q-1} \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} b\big(w_0, u_{j,h}\big) u_{j,h} \right\|_{0,\partial\Omega} \\
&\leq \frac{2C}{\rho \|A_h w_0\|_{0,\partial\Omega}} \ \mu_0 - \mu_{k,h} \ \left(\sum_{j \neq k,k+1,\cdots,k+q-1} b^2\big(w_0, u_{j,h}\big) \right)^{\frac{1}{2}} \\
&\leq \frac{C}{\rho \|A_h w_0\|_{0,\partial\Omega}} |\mu_0 - \mu_{k,h}| \left\| w_0 - \sum_{j=k}^{k+q-1} b(w_0, u_{j,h}) u_{j,h} \right\|_{0,\partial\Omega} \\
&= \frac{C}{\rho \|A_h w_0\|_{0,\partial\Omega}} \ \mu_0 - \mu_{k,h} \ \inf_{v \in M_h(\lambda)} \|w_0 - v\|_{0,\partial\Omega} \\
&\leq \frac{C}{\rho \|A_h w_0\|_{0,\partial\Omega}} \ \mu_0 - \mu_{k,h} \ \operatorname{dist}\big(w_0, M_h(\lambda)\big)
\end{aligned}
$$

Taking the norm on both sides of (4.3), and noting that $u_0 = \frac{A_h w_0}{\|A_h w_0\|_{0,\partial\Omega}}$, the condition $(C1)$ and (4.5), we get

$$
\begin{aligned}
&\| (\mu_0 - \mu_{k,h}) u' \|_{0,\partial\Omega} = \| \sum_{j=1}^{d} \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} b(u_0, u_{j,h}) u_{j,h} \|_{0,\partial\Omega} \\
&= \frac{1}{\| A_h w_0 \|_{0,\partial\Omega}} \left(\sum_{j=1}^{d} \left(\frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} b^2(w_0, \mu_{j,h} u_{j,h}) \right) \right)^{\frac{1}{2}} \\
&\geq \frac{1}{\| A_h w_0 \|_{0,\partial\Omega}} \min_{k \leq j \leq k+q-1} \left| \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} \right| \left(\sum_{j=k}^{k+q-1} b^2(w_0, u_{j,h}) \right)^{\frac{1}{2}} \qquad (4.7) \\
&= \frac{1}{\| A_h w_0 \|_{0,\partial\Omega}} \min_{k \leq j \leq k+q-1} \left| \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} \right| \left\| w_0 - \left(w_0 - \sum_{j=k}^{k+q-1} b(w_0, u_{j,h}) u_{j,h} \right) \right\|_{0,\partial\Omega} \\
&\geq \frac{1}{2 \| A_h w_0 \|_{0,\partial\Omega}} \min_{k \leq j \leq k+q-1} \left| \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} \right|
\end{aligned}
$$

From (4.6) and (4.7) we have

$$
\begin{aligned}
&\operatorname{dist}(u^h, M_h(\lambda)) = \operatorname{dist}\big(\operatorname{sign}\big(\mu_0 - \mu_{k,h}\big) u^h, M_h(\lambda)\big) \\
&\leq \left\| \operatorname{sign}\big(\mu_0 - \mu_{k,h}\big) u^h - \frac{1}{\big\|\big(\mu_0 - \mu_{k,h}\big) u'\big\|_{0,\partial\Omega}} \sum_{j=k}^{k+q-1} \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} b\big(u_0, u_{j,h}\big) u_{j,h} \right\|_h \\
&= \left\| \frac{\big(\mu_0 - \mu_{k,h}\big) u'}{\big\|\big(\mu_0 - \mu_{k,h}\big) u'\big\|_{0,\partial\Omega}} - \frac{1}{\big\|\big(\mu_0 - \mu_{k,h}\big) u'\big\|_{0,\partial\Omega}} \sum_{j=k}^{k+q-1} \frac{\mu_0 - \mu_{k,h}}{\mu_0 - \mu_{j,h}} b\big(u_0, u_{j,h}\big) u_{j,h} \right\|_h
\end{aligned}
$$

$$\leq 2 \parallel A_h w_0 \parallel_{0,\partial\Omega} \max_{k\leq j\leq k+q-1} \left|\frac{\mu_0-\mu_{j,h}}{\mu_0-\mu_{k,h}}\right| \left\|(\mu_0-\mu_{k,h})u' - \sum_{j=k}^{k+q-1} \frac{\mu_0-\mu_{k,h}}{\mu_0-\mu_{j,h}} b(u_0,u_{j,h})u_{j,h}\right\|_h$$

$$\leq \frac{C}{\rho} \max_{k\leq j\leq k+q-1} \mu_0-\mu_{j,h} \ \mathrm{dist}(w_0, M_h(\lambda))$$

The proof is completed.

Next, we shall use Lemma 4.1 and Theorem 2.2 to analyze the error of two-grid discretization Scheme 3.1. We first consider the case of $l = 1$. Denote $H = h_0$, $h = h_1$.

Theorem 4.1. Suppose that $M(\lambda_k) \subset H^{1+s}(\Omega)(s \geq r)$, and $t = min\{m, s\}$. Let (λ_k^h, u_k^h) be an approximate eigenpair obtained by Scheme 3.1($l = 1$) and H is sufficiently small, then there exists $u_k \in M(\lambda_k)$ such that

$$\|u_k^h - u_k\|_h \leqslant C(H^{3t} + h^t) \tag{4.8}$$

$$\|u_k^h - u_k\|_{0,\partial\Omega} \leqslant C(H^{3t} + h^{t+r}) \tag{4.9}$$

$$\lambda_k^h - \lambda_k \ \leqslant C(H^{3t} + h^t)^2 \tag{4.10}$$

Proof. We use Lemma 4.1 to complete the proof. Select $\mu_0 = \frac{1}{\lambda_H}$, $w_0 = u_H$ and $u_0 = \frac{A_h u_H}{\|A_h u_H\|_{0,\partial\Omega}}$. From (2.31) we know that there exists $\bar{u} \in M(\lambda_k)$ such that

$$\parallel \bar{u} - u_H \parallel_h \leq CH^t$$

Using the triangle inequality and (2.31) we can deduce that

$$\begin{aligned} dist(u_H, M_h(\lambda_k)) &\leq \parallel u_H - \bar{u} \parallel_H + dist(\bar{u}, M_h(\lambda_k)) \\ &\leq C(H^t + h^t) \leq CH^t \end{aligned} \tag{4.11}$$

therefore,

$$\inf_{v\in M_h(\lambda_k)} \parallel u_H - v \parallel_{0,\partial\Omega} \leq CH^t$$

when H is small enough, the condition $(C1)$ in Lemma 4.1 is valid.
From (2.29) we can deduce that

$$\mu_0 - \mu_j = \frac{|\lambda_H - \lambda_k|}{|\lambda_H \lambda_k|} \leq CH^{2t} \leq \frac{\rho}{4}$$

$$\mu_j - \mu_{j,h} = \frac{\lambda_{j,h} - \lambda_j}{\lambda_{j,h}\lambda_j} \leq Ch^{2t} \leq \frac{\rho}{4}, j = k-1, k, \dots, k+q, j \neq 0$$

that is, the condition $(C2)$ in Lemmma 4.1 holds.

By (2.14) we see that Step 3 in Scheme 3.1 is equivalent to the following:

$$a_h(u', v) - \lambda_H a_h(A_h u', v) = a_h(A_h u_H, v) \forall v \in S^h$$

$u_k^h = \frac{u'}{\|u'\|_{0,\partial\Omega}}$, i.e.,

$$(\lambda_H^{-1} - A_h)u' = \lambda_H^{-1} A_h u_H, \qquad u_k^h = \frac{u'}{\parallel u' \parallel_{0,\partial\Omega}}$$

Note that $\lambda_H^{-1} A_h u_H$ and u_0 differ by only one constant, then, Step 3 in Scheme 3.1 is equivalent to

$$(\lambda_H^{-1} - A_h)u' = u_0, u_k^h = \frac{u'}{\parallel u' \parallel_{0,\partial\Omega}}$$

From the above arguments we see that the conditions of Lemma 4.1 hold.

Since $M_h(\lambda_k)$ is a q-dimensional space, there must exist $u^* \in M_h(\lambda_k)$ such that

$$\parallel u_k^h - u^* \parallel_h = \mathrm{dist}(u_k^h, M_h(\lambda_k))$$

For $j = k, k+1, \dots, k+q-1$, according to (2.32) we get

$$\begin{aligned} | \mu_0 - \mu_{j,h} | &= \left|\frac{1}{\lambda_H} - \frac{1}{\lambda_{j,h}}\right| \leq \frac{\lambda_H - \lambda_{j,h}}{\lambda_H \lambda_{j,h}} \\ &\leq C\big(|\lambda_H - \lambda_k| + \ \lambda_k - \lambda_{j,h} \ \big) \leq CH^{2t} \end{aligned} \tag{4.12}$$

Therefore, from Lemma 4.1, (4.11) and (4.12) we can deduce that

$$\begin{aligned} \parallel u_k^h - u^* \parallel_h &= \mathrm{dist}\big(u_k^h, M_h(\lambda_k)\big) \\ &\leq \frac{4}{\rho} \max_{k\leq j\leq k+q-1} \mu_0 - \mu_{j,h} \ \mathrm{dist}(u_H, M_h(\lambda_k)) \leq CH^{3t} \end{aligned} \tag{4.13}$$

From (2.31) we know that there exists $u_k \in M(\lambda_k)$, such that $\parallel u^* - u_k \parallel_h = \mathrm{dist}(u^*, M(\lambda_k))$, and

$$||u^* - u_k||_h \leq Ch^t$$

then

$$\parallel u_k^h - u_k \parallel_h \leq \parallel u_k^h - u^* \parallel_h + \parallel u^* - u_k \parallel_h \leq C(H^{3t} + h^t)$$

that is (4.8).

Next, we will prove (4.9). From (2.28) we can deduce that

$$\| u^* - u_k \|_{0,\partial\Omega} \leq Ch^{t+r}$$

which together with (4.13) yields

$$\| u_k^h - u_k \|_{0,\partial\Omega} \leq \| u_k^h - u^* \|_{0,\partial\Omega} + \| u^* - u_k \|_{0,\partial\Omega} \leq C(H^{3t} + h^{t+r})$$

Finally, we use Lemma 3.1 to derive (4.10). From Step 4 of Scheme 3.1, Lemma 3.1, (4.8) and (4.9) we can deduce that

$$\begin{aligned} |\lambda_k^h - \lambda_k| &= \left| \frac{a_h(u_k^h - u_k, u_k^h - u_k)}{\| u_k^h \|_b^2} - \lambda_k \frac{b(u_k^h - u_k, u_k^h - u_k)}{\| u_k^h \|_b^2} \right| \\ &\leq C\|u_k^h - u_k \|_h^2 + |\lambda_k| \, \| u_k^h - u_k \|_{0,\partial\Omega}^2 \big) \\ &\leq 2C(H^{3t} + h^t)^2 \end{aligned}$$

The proof is completed.

V. NUMERICAL EXPERIMENTS

In this section, we will report some numerical experiments for Scheme 3.1 to validate our efficiency of the DG-multigrid method for solving the Steklov eigenvalue problem. We use MATLAB 2017a to solve Our program are compiled under the package of Chen [33]. The test domains are set to be the unit square $\Omega_S := (0,1)^2$ with vertices are (0,1), (1,0), (0,0), (1,1) and the L-shaped domain $\Omega_L := [0,1] \times [0, \frac{1}{2}] \cup [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$, respectively. The numerical results are listed in Table 1 and Table 2, respectively. In Table 1 and Table 2, h stands for the mesh size. And the four smallest approximate eigenvalues on Ω_S are

$$\lambda_1 \approx 0.240079085421, \qquad \lambda_2 \approx 1.492303134531$$
$$\lambda_3 \approx 1.492303134531, \qquad \lambda_4 \approx 2.082647054031$$

The four smallest approximate eigenvalues on Ω_L are

$$\lambda_1 \approx 0.182964236872, \qquad \lambda_2 \approx 0.893672918808$$
$$\lambda_3 \approx 1.688600483582, \qquad \lambda_4 \approx 3.217859788054$$

$\lambda_{k,h}$: The kth eigenvalue of (2.2) obtained by directly solving using the $eigs$ command on the grid $\mathcal{T}_H$;
λ_k^h : The kth eigenvalue derived from Scheme 3.1;
CPU_h (s): The CPU time (s) used to solve the eigenvalue problem directly on the fine grid $\mathcal{T}_h$;
CPU^h (s): To calculate the CPU time (s) from the program started running to the current using Scheme 3.1.

This paper presents a study on the two-grid discretization of Steklov eigenvalue problems using the discontinuous Galerkin method. Based on our approach, we solve the eigenvalue problem on the fine grid $\mathcal{T}_h$ using linear elements and also provide solutions using Scheme 3.1 Numerical experiments are conducted on Ω_S and Ω_L. From Table 1 and Table 2, it can be seen that when the mesh size increases, the advantages of the two-grid discretization method with shifted inverse iteration become more apparent, indicating the efficiency of our approach. That is, comparing to directly solving the eigenvalue problem on the fine grid, the two-grid discretization method based on shifted inverse iteration requires less CPU time. Therefore, this method has strong practical value for solving Steklov eigenvalue problems.

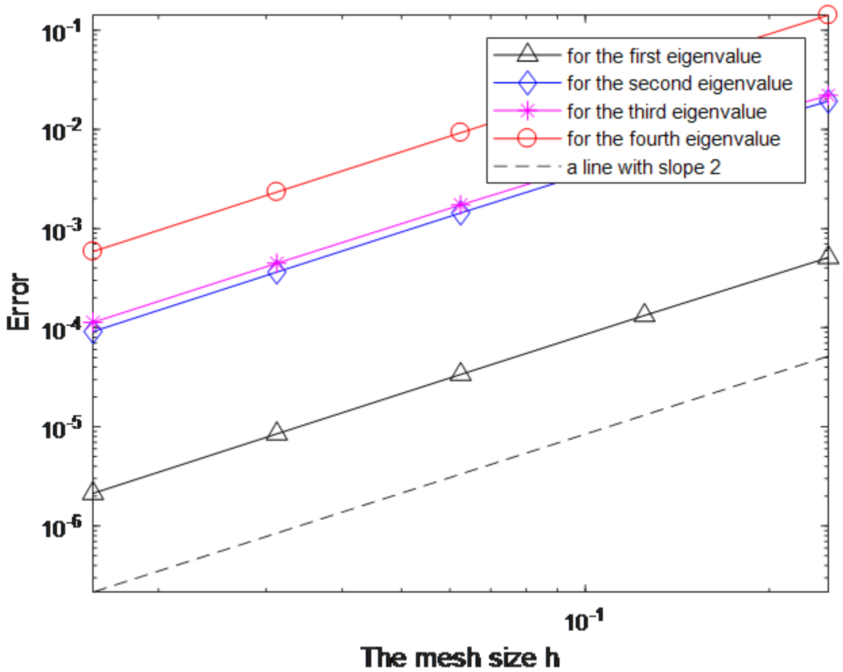

Figure 1: The error curves of the approximation for the first fourth eigenvalues of (2.2) obtained by solving on linear element Ω_S.

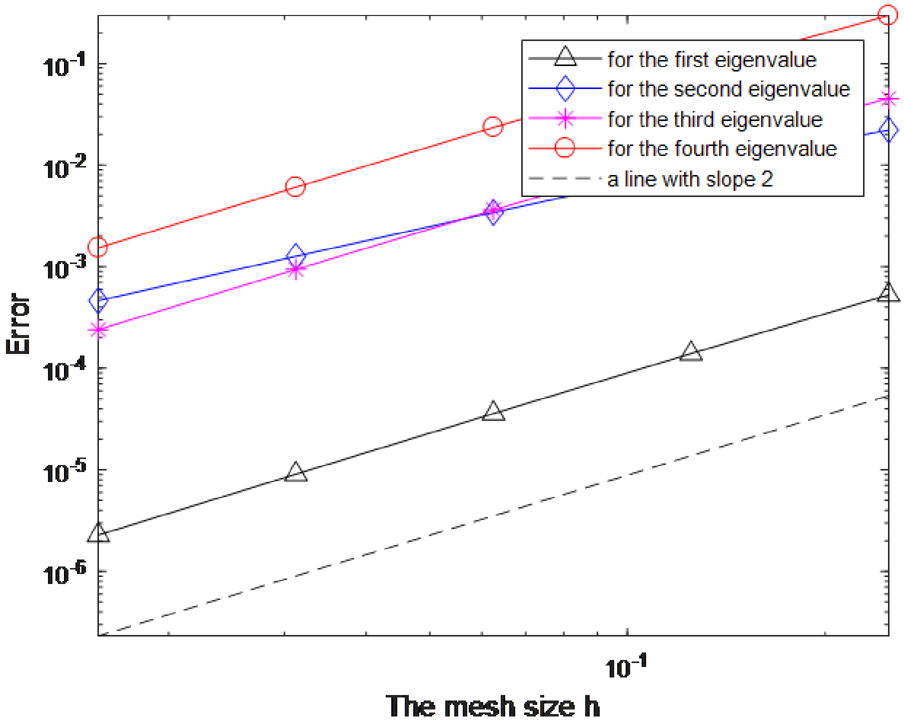

Figure 2: The error curves of the approximation for the first fourth eigenvalues of (2.2) obtained by solving on linear element Ω_L.

Table 1: The first fourth eigenvalues of (2.1) solved using linear elements on domain Ω_S, based on scheme 3.1.

k	H	h	$\lambda_{k,H}$	$\lambda_{k,h}$	λ_k^h	CPU_h	CPU^h
1	$\sqrt{2}/8$	$\sqrt{2}/64$	0.240211716943679	0.240081216887906	0.240087584025356	0.53	0.03
1	$\sqrt{2}/16$	$\sqrt{2}/128$	0.240112826863705	0.240079619065521	0.240087583647472	2.34	0.03

1	$\sqrt{2}/32$	$\sqrt{2}/256$	0.240087583631145	0.240079218879418	0.240087583631562	11.05	0.07
2	$\sqrt{2}/8$	$\sqrt{2}/64$	1.497796681403516	1.492394549965922	1.492682256586994	0.54	0.02
2	$\sqrt{2}/16$	$\sqrt{2}/128$	1.493737311068824	1.492326033500083	1.492669551372664	2.30	0.03
2	$\sqrt{2}/32$	$\sqrt{2}/256$	1.492666919245250	1.492308863965837	1.492666919245329	11.20	0.07
3	$\sqrt{2}/8$	$\sqrt{2}/64$	1.497796681403516	1.492394549965922	1.492682256586994	0.54	0.02
3	$\sqrt{2}/16$	$\sqrt{2}/128$	1.493737311068824	1.492326033500083	1.492669551372664	2.30	0.03
3	$\sqrt{2}/32$	$\sqrt{2}/256$	1.492666919245250	1.492308863965837	1.492666919245329	11.20	0.07
4	$\sqrt{2}/8$	$\sqrt{2}/64$	2.119021802190831	2.083232720433984	2.081623917337848	0.53	0.03
4	$\sqrt{2}/16$	$\sqrt{2}/128$	2.091898207310220	2.082793751735617	2.084837094486877	2.32	0.03
4	$\sqrt{2}/32$	$\sqrt{2}/256$	2.084980101241345	2.082683761801382	2.084980101242731	11.14	0.07

Table 2: The first fourth eigenvalues of (2.1) solved using linear elements on domain Ω_L, based on scheme 3.1.

k	H	h	$\lambda_{k,H}$	$\lambda_{k,h}$	λ_k^h	CPU_h	CPU^h
1	$\sqrt{2}/8$	$\sqrt{2}/64$	0.183103578919708	0.182966511922233	0.182973282940761	0.37	0.02
1	$\sqrt{2}/16$	$\sqrt{2}/128$	0.182999984078493	0.182964807413117	0.182973282661870	1.66	0.02
1	$\sqrt{2}/32$	$\sqrt{2}/256$	0.182973282649468	0.182964379709057	0.182973282649819	7.55	0.05
2	$\sqrt{2}/8$	$\sqrt{2}/64$	0.902722938388933	0.894134444101383	0.894965363626095	0.36	0.02
2	$\sqrt{2}/16$	$\sqrt{2}/128$	0.897087899512647	0.893832650873214	0.894941335725347	1.69	0.02
2	$\sqrt{2}/32$	$\sqrt{2}/256$	0.894937883634208	0.893717985383333	0.894937883634200	7.67	0.05
3	$\sqrt{2}/8$	$\sqrt{2}/64$	1.701946204490103	1.688840311680806	1.665061852769322	0.36	0.02
3	$\sqrt{2}/16$	$\sqrt{2}/128$	1.692214637235342	1.688661107773249	1.689125207477825	1.66	0.02
3	$\sqrt{2}/32$	$\sqrt{2}/256$	1.689540378763166	1.688615730973038	1.689540378764314	7.68	0.04
4	$\sqrt{2}/8$	$\sqrt{2}/64$	3.304885179800326	3.219389751051097	3.223432266030927	0.40	0.02
4	$\sqrt{2}/16$	$\sqrt{2}/128$	3.241311541143945	3.218244256049463	3.223895724733414	1.69	0.02
4	$\sqrt{2}/32$	$\sqrt{2}/256$	3.223904609238326	3.217956122839084	3.223904609239786	7.72	0.04

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