International Journal of Engineering Science Invention
ISSN (Online): 2319 – 6734, ISSN (Print): 2319 – 6726
www.ijesi.org Volume 3 Issue 8 ‖ August 2014 ‖ PP.11-16

k - Symmetric Doubly Stochastic, s - Symmetric Doubly Stochastic and s – k - Symmetric Doubly Stochastic Matrices

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ABSTRACT: *The basic concepts and theorems of k-symmetric doubly stochastic s-symmetric doubly stochastic and s-k-symmetric doubly stochastic matrices are introduced with examples.*

KEY WORDS: *k-symmetric doubly stochastic matrix, s-symmetric doubly stochastic matrix and s-k-symmetric doubly stochastic matrix.*

AMS CLASSIFICATIONS: 15A51, 15B99

I. INTRODUCTION

We have already seen the concept of symmetric doubly stochastic matrices. In this paper the symmetric doubly stochastic matrix is developed in real matrices. Recently Hill and Waters[2] have developed a theory of k-real matrices as a generalization of s-real matrices. Ann Lee[1] has initiated the study of secondary symmetric matrices, that is matrices whose entries are symmetric about the secondary diagonal. Ann Lee[1] has shown that the matrix A, the usual transpose A^T and secondary transpose A^S are related as $A^S = VA^TV$ and $A^T = VA^SV$ where V is a permutation matrix with units in the secondary diagonal.

II. PRELIMINARIES AND NOTATIONS

A^T - Transpose of A
Let k be a fixed product of disjoint transpositions in S_n and 'K' be the permutation matrix associated with k.
Clearly K satisfies the following properties. $K^2 = I$, $K^T = K$.

III. DEFINITIONS AND THEOREMS

DEFINITION: 1

A matrix $A \in R^{n \times n}$ is said to be symmetric doubly stochastic matrix if $A = A^T$ and

$\sum_{i=1}^{n} a_{ij} = 1$, j = 1, 2, ………n

and $\sum_{j=1}^{n} a_{ij} = 1$, i = 1, 2, ………n and all $a_{ij} \geq 0$.

If A is doubly stochastic and also symmetric then it is called a symmetric doubly stochastic matrix.

DEFINITION: 2

A matrix $A \in R^{n \times n}$ is said to be k-symmetric doubly stochastic matrix if $A = K A^T K$

THEOREM: 1

Let $A \in R^{n \times n}$ is k-symmetric doubly stochastic matrix then $A = K A^T K$.

Proof:

$$\begin{aligned} K A^T K &= KAK \text{ where } A^T = A \\ &= AKK \text{ where } AK = KA \\ &= AK^2 = A \text{ where } K^2 = I \end{aligned}$$

THEOREM: 2

Let $A^T \in R^{n \times n}$ is k-symmetric doubly stochastic matrix then $A^T = KAK$.

Proof:

$$\begin{aligned} K A K &= KA^TK \text{ where } A = A^T \\ &= A^TKK \text{ where } K A^T = A^T K \\ &= A^T K^2 = A^T \text{ where } K^2 = I \end{aligned}$$

THEOREM: 3

Let A, B $\in R^{n \times n}$ is k-symmetric doubly stochastic matrix then $\frac{1}{2}$(A + B) is k-symmetric doubly stochastic matrix.

Proof:

Let A and B are k-symmetric doubly stochastic matrix if $A = K A^T K$ and $B = KB^TK$.

To prove $\frac{1}{2}$(A + B) is k-symmetric doubly stochastic matrix we will show that

$$\frac{1}{2}(A + B) = K \frac{1}{2}(A + B)^T K$$

Now $K \frac{1}{2}(A + B)^T K = K \frac{1}{2}(A^T + B^T) K = \frac{1}{2} K (A^T + B^T) K = \frac{1}{2} (KA^T + KB^T) K = \frac{1}{2} (KA^T K + KB^T K)$

$= \frac{1}{2}(A + B)$ where $K A^T K = A$ and $KB^TK = B$

THEOREM: 4

Any k-symmetric doubly stochastic matrix can be represent as sum of k-symmetric doubly stochastic matrix and skew k-symmetric doubly stochastic matrix.

Proof:

To prove that $\frac{1}{2}(A + KA^TK)$ and $\frac{1}{2}(A - KA^TK)$ are k-symmetric doubly stochastic matrices the we will show that $\frac{1}{2}(A + KA^TK) = K \frac{1}{2}(A + KA^TK)^T K$ and $\frac{1}{2}(A - KA^TK) = K \frac{1}{2}(A - KA^TK)^T K$.

$K \frac{1}{2}(A + KA^TK)^T K = \frac{1}{2}(A + KA^TK)$ using theorem 3 and $K \frac{1}{2}(A - KA^TK)^T K = \frac{1}{2}(A - KA^TK)$.

Then $\frac{1}{2}(A + KA^TK) + \frac{1}{2}(A - KA^TK) = 2A/2 = A$. Hence the theorem is proved.

THEOREM: 5

If A and B are k-symmetric doubly stochastic matrices then AB is also k-symmetric doubly stochastic matrix.

Proof:

Let A and B are k-symmetric doubly stochastic matrix if $A = K A^T K$ and $B = KB^TK$.

Since A^T and B^T are also k-symmetric doubly stochastic matrices then $A^T = KAK$ and $B^T = KBK$.

To prove A B is k-symmetric doubly stochastic matrix we will show that

$AB= K (A B)^T K$

Now $K (A B)^T K= KB^TA^TK = K(KBK)(KAK)K$ where $A^T = KAK$ and $B^T = KBK$.

$= K^2B K^2A K^2 = BA$ where $K^2 = I$

$= AB$ where $BA = AB$

THEOREM: 6

If A and B are k-symmetric doubly stochastic matrices and K is the permutation matrix, k = {(1), (2 3)} then KA is also k-symmetric doubly stochastic matrix.

Proof:

Let A and B are k-symmetric doubly stochastic matrix if $A = K A^T K$ and $B = KB^TK$.

Since A^T and B^T are also k-symmetric doubly stochastic matrices then $A^T = KAK$ and $B^T = KBK$.

To prove K B is K-symmetric doubly stochastic matrix we will show that $KA= K (KA)^T K$

Now $K (KA)^T K= K(A^TK^T)K = KA^TK^TK = KA^T$ where $K^T K = I$.

$= KA$ where $KA^T = KA$

RESULT:

For $A \in R^{n \times n}$ is symmetric doubly stochastic matrices for the following are holds.

[1] $A = K A^T K$

[2] KA is symmetric doubly stochastic matrix.

[3] AK is symmetric doubly stochastic matrix.

Example:

$$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \quad \text{and } k = (1)(2\ \ 3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(i) $K A^T K = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1/2&1/2\\1/2&0&1/2\\1/2&1/2&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} = A$

Similarly $KAK = A^T$

(ii) $KA = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1/2&1/2\\1/2&0&1/2\\1/2&1/2&0\end{pmatrix} = \begin{pmatrix}0&1/2&1/2\\1/2&1/2&0\\1/2&0&1/2\end{pmatrix} = (KA)^T$

$\Rightarrow$ KA is symmetric doubly stochastic matrix.
Similarly KA^T is also symmetric doubly stochastic matrix.

(iii) $AK = \begin{pmatrix}0&1/2&1/2\\1/2&0&1/2\\1/2&1/2&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} = \begin{pmatrix}0&1/2&1/2\\1/2&1/2&0\\1/2&0&1/2\end{pmatrix} = (AK)^T$

$\Rightarrow$ AK is symmetric doubly stochastic matrix.
Similarly $A^T K$ is also symmetric doubly stochastic matrix.

DEFINITION: 3

A matrix $A \in R^{n \times n}$ is said to be s-symmetric doubly stochastic matrix if $A^S = V A^T V$ where V is a permutation matrix with units in the secondary diagonal.

THEOREM: 7

Let $A \in R^{n \times n}$ is s-symmetric doubly stochastic matrix then $A^S = V A^T V$.

Proof:

$$V A^T V = V (VA^S V) V = \mathbf{V^2} A^S V^2$$
$$= A^S \text{ where } V^2 = I$$

THEOREM: 8

Let $A^T \in R^{n \times n}$ is s-symmetric doubly stochastic matrix then $A^T = VA^SV$.

Proof:

$$V A^S V = V (VA^T V) V = \mathbf{V^2} A^T V^2$$
$$= A^T \text{ where } V^2 = I$$

THEOREM: 9

Let $A, B \in R^{n \times n}$ is s-symmetric doubly stochastic matrix then $\frac{1}{2}(A + B)$ is s-symmetric doubly stochastic matrix.

Proof:

Let A and B are s-symmetric doubly stochastic matrices if $A^S = V A^T V$ and $B^S = VB^TV$.
To prove $\frac{1}{2}(A + B)$ is s-symmetric doubly stochastic matrix we will show that

$\frac{1}{2}(A + B)^S = V \frac{1}{2}(A + B)^T V$

Now $V \frac{1}{2}(A + B)^T V = V \frac{1}{2}(A^T + B^T) V = \frac{1}{2} V(A^T + B^T) V = \frac{1}{2}(VA^T + VB^T) V = \frac{1}{2}(VA^T V + VB^T V)$

$= \frac{1}{2}(A^S + B^S)$ where $V A^T V = A^S$ and $VB^TV = B^S$

$= \frac{1}{2}(A + B)^S$

THEOREM: 10

If A and B are s-symmetric doubly stochastic matrices then AB is also s-symmetric doubly stochastic matrix.

Proof:

Let A and B are s-symmetric doubly stochastic matrices if $A^S = V A^T V$ and $B^S = VB^TV$.
Since A^T and B^T are also s-symmetric doubly stochastic matrices then $A^T = VA^SV$ and $B^T = VB^SV$.
To prove A B is s-symmetric doubly stochastic matrix we will show that

$(AB)^S = V(A B)^T V$

Now $V (A B)^T V = VB^TA^TV = V(VB^S V)(VA^SV)V$ where $A^T = KA^SK$ and $B^T = KB^SK$.

$= V^2 B^SV^2A^S V^2 = B^S A^S$ where $V^2 = I$

$= (AB)^S$

THEOREM: 11

If A is s-symmetric doubly stochastic matrix and V is a permutation matrix with units in the secondary diagonal then VA is also s-symmetric doubly stochastic matrix.

Proof:

Let A is s-symmetric doubly stochastic matrices if $A^S = V A^T V$. Since A^T is s-symmetric doubly stochastic matrices then $A^T = VA^SV$. To prove VA is s-symmetric doubly stochastic matrix we will show that

$(VA)^S = V(VA)^T V$

Now $V (VA)^T V = V(A^TV^T)V = V(VA^SV) V^2$

$= A^S V^S$ where $V^2 = I$

$= (VA)^S$

RESULT:

For $A \in R^{n \times n}$ is s-symmetric doubly stochastic matrix for the following are holds.

[1] $A^S = V A^T V$

[2] VA is symmetric doubly stochastic matrix.

[3] AV is symmetric doubly stochastic matrix.

Example:

$$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \qquad A^T = A^S = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(i) $$V A^T V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = A^S$$

Similarly $VA^SV = A^T$

(ii) $$VA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} = (VA)^T$$

$\Rightarrow$ VA is symmetric doubly stochastic matrix.

Similarly VA^T is also symmetric doubly stochastic matrix.

(iii) $$AV = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} = (AV)^T$$

$\Rightarrow$ AV is symmetric doubly stochastic matrix.

Similarly $A^T V$ is also symmetric doubly stochastic matrix.

DEFINITION: 4

A matrix $A \in R^{n \times n}$ is said to be s-k-symmetric doubly stochastic matrix if

[1] $A = KVA^TVK$

[2] $A^T = KVAVK$

[3] $A = VKA^TKV$

[4] $A^T = VKAKV$

Where V is a permutation matrix with units in the secondary diagonal and K is a permutation matrix and k = {(1) (2 3)}.

THEOREM: 12

Let $A \in R^{n \times n}$ is s-k-symmetric doubly stochastic matrix then

[1] $A^S = KVA^TVK$

[2] $A^T = KVA^SVK$

[3] $A^S = VKA^TKV$

[4] $A^T = VK A^S KV$

Proof:

(i) $KVA^TVK = K(VA^TV)K$
$= KA^SK$ where $VA^TV = A^S$
$= K(A^T)^TK = K A^TK$ where $A^T = A$
$= A = A^S$ where $KA^TK = A$ and $A = A^S$

(ii) $KVA^SVK = K(VA^S V)K = K A^TK$ where $VA^SV = A^T$
$= \mathbf{A}$ where $K A^TK = A$
$= A^T$ where $A = A^T$

(iii) $VKA^TKV = V(KA^TK)V$
$= VAV$ where $KA^TK = A$
$= VA^TV$ where $A = A^T$
$= A^S$ where $VA^TV = A^S$

(iv) $VKA^SKV = V(KA^SK)V = V(K(A^T)^TK)V$
$= VA^TV$ where $K(A^T)^TK = A^T$
$= A^S$ where $V A^T V= A^S$
$= (A^T)^T = A^T$ where $A = A^T$

THEOREM: 13

Let $A, B \in R^{n \times n}$ is s-k-symmetric doubly stochastic matrix then $\frac{1}{2}(A + B)$ is s-k-symmetric doubly stochastic matrix.

Proof:

Let A and B are s-k-symmetric doubly stochastic matrix if $A = KV A^T VK$ and $B = KVB^TVK$.

To prove $\frac{1}{2}(A + B)$ is s-k-symmetric doubly stochastic matrix we will show that

$\frac{1}{2}(A + B) = KV \frac{1}{2}(A + B)^T VK$

Now $KV \frac{1}{2}(A + B)^T VK = K(V\frac{1}{2}(A + B)^T V)K = K\frac{1}{2}(A + B)^S K$ using theorem (9)

$= K\frac{1}{2}(A + B)^T K = \frac{1}{2}(A + B)$ using theorem (3)

THEOREM: 14

If A and B are s-k-symmetric doubly stochastic matrix then AB is also s-k-symmetric doubly stochastic matrix.

Proof:

Let A and B are s-k-symmetric doubly stochastic matrix if $A = KV A^T VK$ and $B = KVB^TVK$.
Since A^T and B^T are also s-k-symmetric doubly stochastic matrices $A^T = KVAVK$ and $B^T = KVBVK$.
To prove A B is s-k-symmetric doubly stochastic matrix we will show that

$AB = KV(AB)^T VK$

Now $KV(AB)^T VK = K(V(AB)^T)VK = K(AB)^SK$ using theorem (10)

$= K(AB)^TK = AB$ using theorem (5)

RESULT:

For $A \in R^{n \times n}$ is s-k-symmetric doubly stochastic matrix the following are equivalent.

(i) $A = KVA^TVK$ (ii) $A^T = KVAVK$ (iii) $A = VKA^TKV$ (iv) $A^T = VKAKV$

Example:

$$A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \quad K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(i) $$KVA^TVK = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = A$$

(ii) $$KVAVK = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = A^T$$

(iii) $VKA^TKV = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = A$

(iv) $VKAKV = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = A^T$

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