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Existence of Solutions of Fractional Neutral Integrodifferential Equations with Infinite Delay

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ABSTRACT : *In this paper, we study the existence of mild solutions for nonlocal Cauchy problem for fractional neutral nonlinear integrodifferential equations with infinite delay. The results are obtained by using the Banach contraction principle. Finally, an application is given to illustrate the theory.*

KEYWORDS : *Fractional neutral evolution equations, nonlocal Cauchy problem, mild solutions, analytic semigroup, Laplace transform probability density.*

I. INTRODUCTION

In this article, we study the existence of mild solutions for nonlocal Cauchy problem for fractional neutral integro evolution equations with infinite delay in the form

$$^{C}D_t^q(x(t)+f(t,x_t,\int_0^t h(s,x(s))ds)=Ax(t)+g(t,x_t,\int_0^t h(s,x(s))ds)\,,t\in[0,b] \text{ -------}\rightarrow(1)$$

$$x_0=\varphi+q(x_{t_1},x_{t_2},.......x_{t_n})\in \text{B--------}\rightarrow(2)$$

$^{C}D_t^q$ is the Caputo fractional derivative of order 0 < q < 1, A is the infinitesimal generator of an analytic semigroup of bounded linear operators T(t) on a Banach space X. The history $x_t:(-\infty,0]\rightarrow X$ given by $x_t(\theta)=x(t+\theta)$ belongs to some abstract phase space **B** defined axiomatically, $0 < t_1 < t_2 \ldots < t_n \leq b$, q : B-→B and $f,g:[0,b]$ x B -→ X are appropriate functions. Fractional differential equations is a generalization of ordinary differential equations and integration to arbitrary non – integer orders. Recently, fractional differential equations is emerging as an important area of investigation in comparsion with corresponding theory of classical differential equations. It is an alternative model to the classical nonlinear differential models. It is widely and efficiently used to describe many phenomena in various fieldsof engineering and scientific disciplines as the mathematical modeling of systems and processes in many fields, for instance, physics, chemistry, aerodynamic, electrodynamics of complex medium, polymer rheology, viscoelasticity, porous media and so on. There has been a significant development in fractional differential and partial differential equations in recent years; see the monographs of Kilbas et al[13], Miller and Ross[16], Podlubny[20], Lakshmikanthan et al[14]. Recently, some authors focused on fractional functional differential equations in Banach spaces [3,5 - 9, 15,17,18,21 – 23,25 – 32].

There exist an extensive literature of differential equations with nonlocal conditions. Byszewski [1,2] was first formulated and proved the result concerning the existence and uniqueness of mild solutions to abstract Cauchy problems with nonlocal initial conditions. Hernandez [10,11] study the existence of mild, strong and classical solutions for the nonlocal neutral partial functional differential equation with unbounded delay. In [9], Guerekatadiscussed the existence, uniqueness and continuous dependence on initial data of solutions to the nonlocal Cauchy problem

$$\dot{x}(t)=Ax(t)+g(t,x_t)\,,t\in(\sigma,T]$$
$$x_0=\varphi+q(x_{t_1},x_{t_2},.......x_{t_n})$$

where A is the infinitesimal generator of a C_0 - semigroup of linear operators; $t_i \in [\sigma,T]$; $x_t \in C([-r,0]:X)$ and $q: C([-r,0]:X)^n \to X$; $f:[\sigma,T] \times C([-r,0]:X) \to X$ are appropriate functions. Recently, Zhou [31] studied the nonlocal Cauchy problem of the following form

$$ {}^{C}D_t^q(x(t)-h(t,x_t))+Ax(t)=f(t,x_t), t\in[0,b] $$
$$ x_0(v)+g(x_{t_1},x_{t_2},....x_{t_n})(v)=\varphi(v), v\in[-r,0], $$

${}^{C}D^q$ is the Caputo fractional derivative of order $0 < q < 1$, $0 < t_1 <\ldots< t_n < a$, $a > 0$.

A is the infinitesimal generator of an analytic semigroup $T(t)_{t\geq 0}$ of operators on E, $f,h:[0,\infty) \times C \to E$ and $g: C^n \to C$ are given functions satisfying some assumptions, $\varphi \in C$ and define x_t by $x_t(v) = x(t+v)$, for $v \in [-r,0]$.

This paper is organized as follows. In section 2, we recall recent results in the theory of fractional differential equations and introduce some notations, definitions and lemmas which will be used throughout the papers [31,32]. In section 3, we study the existence result for the IVP (1) - (2). The last section is devoted to an example to illustrate the theory.

II. PRELIMINARIES

Throughout this paper, let A be the infinitesimal generator of an analytic semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operators on X. Let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Then for $0<\eta\leq 1$, it is possible to define the fractional power A^η as a closed linear operator on its domain $D(A^\eta)$. For analytic semigroup $\{T(t)\}_{t\geq 0}$, the following properties will be used.

(i) There is a $M \geq 1$, such that

$$ M = \sup_{t\in[0,+\infty)} |T(t)| < \infty $$

(ii) For any $\eta \in (0,1]$, there exists a positive constant C_η such that

$$ |A^\eta T(t)| \leq \frac{C_\eta}{t^\eta}, 0 < t \leq b. $$

We need some basic definitions and properties of the fractional calculus theory which will be used for throughout this paper.

Definition 2.1. The fractional integral of order γ with the lower limit zero for a function f is defined as

$$ I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, t>0, \gamma>0, $$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(.)$ is the gamma function.

Definition 2.2. The Riemann – Liouville derivative of order γ with the lower limit zero for a function $f:[0,\infty) \to R$ can be written as

$$ {}^{L}D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, t>0, n-1<\gamma<n, $$

Definition 2.3. The Caputo derivative of order γ for a function $f:[0,\infty) \to R$ can be written as

$$ {}^{L}D^\gamma f(t) = {}^{L}D^{\gamma} \left(f(t) - \sum_{k=1}^{n-1} \frac{t^k}{k!} f^k(0) \right), t>0, n-1<\gamma<n, $$

Remark 2.4. (i) If $f(t) \in C^n[0,\infty)$, then

$$ {}^{L}D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^n(s)}{(t-s)^{\gamma+1-n}} ds = I^{n-\gamma} f^n(t), t>0, n-1<\gamma<n, $$

(ii)The Caputo derivative of a constant is equal to zero.
(iii) If f is an abstract function with values in X, then integrals which appear in Definition 2.2 and 2.3 are taken in Boehner's sense.

We will herein define the phase space Baxiomatically, using ideas and notation developed in [12]. More precisely, Bwill denote the vector space of functions defined from (-∞,0] into X endowed with a seminorm denoted as $\|.\|_B$and such that the following axioms hold:

(A) If $x:(-\infty,b)\to X$ is continuous on [0,b] and x_0∈**B,** then for every t ∈ [0,b] the following conditions hold:

(i) x_t is inB**.**

(ii) $\| x(t)\| \le H \| x_t \|_B$

(iii) $\|x_t\|_B \le K(t)\sup\{\| x(s)\| : 0\le s\le t\}+M(t)\| x_0 \|_B,$

Where H > 0 is a constant; K,M: [0,∞) →[1,∞)**,** K(.) is continuous, M(.) is locally bounded, and H, K(.), M(.) are independent of x(.).

(A1) For the function x(.) in **(A)**, x_t is a B– valued continuous function on [0,b].

(B) The space Bis complete**.**

Example 2.5.**The Phase Space** $C_r xL^P(h,X)$.

Let r≥ 0, 1≤ $p<\infty$ and h : (-∞,−r] →R be a non – negative, measurable function which satisfies the conditions (g – 5) – (g – 6) in the terminology of [12]. Briefly, this means that g is locally integrable and there exists a non-integer, locally bounded function η(.) on (-∞,0] such that $h(\xi+\theta)\le \eta(\xi)h(\theta)\ for all \xi\le 0$ and $\theta\in(-\infty,-r)\setminus N_\xi$, where $N_\xi \subseteq(-\infty,-r)$ is a set with Lebesgue measure zero. The space $C_r xL^P(h,X)$ consists of all classes of functions $\varphi:(-\infty,0]\to X$ such that φ is continuous on [-r,0] and is Lebesgue measurable, and h||φ||p is Lebesgueintegrable on (-∞,-r). The seminorm in $C_r xL^P(h,X)$ defined by

$$\|\varphi\|_B := \sup\{\| \varphi(\theta)\| : -r\le\theta\le 0\}+\left(\int_{-\infty}^{-r} h(\theta)\| \varphi(\theta)\|^p\, d\theta\right)^{1/p}$$

The space $C_r xL^P(h,X)$ satisfies the axioms (**A**), (**A1**) and (**B**). Moreover, when r =0 and p =2, we can take H=1, $K(t)=1+\left(\int_{-t}^{0} h(\theta)d\theta\right)^{1/2}$ and $M(t)=\eta(-t)^{1/2},\ for t\ge 0$ (see [12, Theorem 1.3.8] for details).

For additional details concerning phase space we refer the reader to [12].

The following lemma will be used in the proof of our main results.

Lemma 2.6.[31, 32] The operators $\Im$ and ζ have the following properties:

(i) For any fixed t≥ 0, $\Im$ (t) and ζ (t) are linear and bounded operators, i.e., for any x∈ X,

$$\| \Im(t)x\| \le M\| x\| \text{and} \| \zeta(t)x\| \le \frac{qM}{\Gamma(1+q)}\| x\| .$$

(ii) $\{\Im(t), t\ge 0\}$ and $\{\zeta(t), t\ge 0\}$ are strongly continuous.

(iii) For every t> 0, $\Im$ (t) and ζ (t) are also compact operators if T(t), t > 0 is compact.

III. EXISTENCE RESULTS

In order to define the concept of mild solution for the system (1.1) - (1.2), by comparison with the fractional differential equations given in [31,32], we associate system (1.1) – (1.2) to the integral equation

$$x(t) = \Im(t)(\varphi(0) + f(0,\varphi) + q(x_{t_1}, x_{t_2}, \ldots. x_{t_n})(0)) - f(t, x_t, \int_0^t h(s, x(s))ds) -$$

$$\int_0^t (t-s)^{q-1} A\zeta(t-s) f(s, x_s, \int_0^s h(t, x(t))dt)ds + \int_0^t (t-s)^{q-1} A\zeta(t-s) g(s, x_s, \int_0^s k(t, x(t))dt)ds \quad \to (3)$$

where

$$\Im(t) = \int_0^\infty \xi_q(\theta) T(t^q\theta)d\theta,\ \zeta(t) = q\int_0^\infty \theta\xi_q(\theta)T(t^q\theta)d\theta,$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-1/q}\varpi_q(\theta^{-1/q}) \geq 0$$

$$\varpi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \theta \in (0,\infty),$$

andξ_q is a probability density function defined on $(0,\infty)$, that is

$$\xi_q(\theta) \geq 0\ \ \theta \in (0,\infty) and \int_0^\infty \xi_q(\theta)d\theta = 1$$

In the sequel we introduce the following assumptions.

(H$_1$) $q : B^n \to B$ is continuous and exist positive constants $L_i(q)$ such that

$$\| q(\psi_1, \psi_2, \ldots, \psi_n) - q(\varphi_1, \varphi_2, \ldots \varphi_n) \| \leq \sum_{i=1}^{n} L_i(q) \| \psi_i - \varphi_i \|_B,$$

for every ψ_i, $\varphi_i \in B_r[0,B]$.

(H$_2$) The function f(.) is $(-A)^\vartheta$– valued , f: I x B x B → $[D((-A)^\vartheta)]$, the functions g(.) is defined on g : I xB xB → X and there exist positive constants L_f and L_g such that for all $t_i, \psi_i, \varphi_k \in I \times B \times B$

$$\| (-A)^\vartheta f(t_1, \psi_1, \phi_1) - (-A)^\vartheta f(t_2, \psi_2, \phi_2) \| \leq L_f (| t_1 - t_2 | + \| \psi_1 - \psi_2 \|_B + \| \phi_1 - \phi_2 \|_B),$$

$$\| g(t_1, \psi_1, \phi_1) - g(t_2, \psi_2, \phi_2) \| \leq L_g (| t_1 - t_2 | + \| \psi_1 - \psi_2 \|_B + \| \phi_1 - \phi_2 \|_B)$$

(H$_3$) Thefunction h,k is defined on h,k : I x B → X and there exist positive constants L_h and L_k such that

$$\| h(t_1, \phi_1) - h(t_2, \phi_2) \| \leq L_h \| \phi_1 - \phi_2 \|$$

$$\| k(t_1, \phi_1) - k(t_2, \phi_2) \| \leq L_k \| \phi_1 - \phi_2 \|$$

Remark 3.1. Throughout this section , M_b and K_b are the constants $M_b = \sup_{s\in[0,b]} M(s), K_b = \sup_{s\in[0,b]} K(s)$ and $N_{(-A)^\vartheta f}$, N_f, N_g, N_h and N_k represent the supreme of the functions $(-A)^\vartheta f$, f and g on [0,b] x $B_r[0,B]$ x X and h, k on [0,b] x $B_r[0,B]$.

Theorem 3.2. Let conditions (H_1) - (H_3) be hold. If

$$\rho = ((M_b + K_b MH)\|\varphi\|_B + (M_b + K_b M)N_q + (M + N_h)K_b N_f + \frac{K_b N_{(-A)^\beta f}\Gamma(1+\beta)C_{1-\beta}b^{q\beta}}{\beta\Gamma(1+\beta q)} + \frac{K_b N_g N_k M_q}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)}) < r \quad \text{and}$$

$$A = \max\left\{ M_b\left(M_b \sum_{i=1}^{n} L_i(q) + K_b\theta \right), K_b\left(M_b \sum_{i=1}^{n} L_i(q) + K_b\theta \right) \right\} < 1$$

where

$$\theta = \left[M\sum_{i=1}^{n} L_i(q) + L_f\left((M+1)\|(-A)^{-\vartheta}\| + \frac{\Gamma(1+\beta)C_{1-\beta}b^{q\beta}}{\beta\Gamma(1+\beta q)} \right) + \frac{Mq}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)} \right]$$

Then there exists a mild solution of the system (1) – (2) on I.

Proof. Consider the space $S(b) = \{x : (-\infty, b] \to X : x_0 \in B; x \in C([0,b]:X)\}$ endowed with the norm

$$\|x\|_{S(b)} := M_b\|x_0\|_B + K_b\|x\|_b$$

Let Y=B_r[0,S(b)], we define the operator Γ:Y → S(b) by

$$\Gamma x(t) = \Im(t)(\varphi(0) + f(0,\varphi) + q(x_{t_1}, x_{t_2}, \ldots. x_{t_n})(0)) - f(t, x_t, \int_0^t h(s, x(s))ds)$$

$$-\int_0^t (t-s)^{q-1} A\zeta(t-s) f(s, x_s, \int_0^s h(t, x(t))dt)ds + \int_0^t (t-s)^{q-1}\zeta(t-s) g(s, x_s, \int_0^s k(t, x(t))dt)ds,$$

$(\Gamma u)_0 = \varphi + q(x_{t_1}, x_{t_2}, \ldots. x_{t_n})$, for t ϵ [0,b].

Using an similar argument on the proof of Theorem 3.1 in [10], we will prove that the Γ is continuous. Next we will prove that Γ(Y) ⊂ Y.

Direct calculation gives that $(t-s)^{q-1} \in L^{\frac{1}{1-q_1}}[0,t]$, for t ϵ J and q_1ϵ [0,q). Let $a = \frac{q-1}{1-q_1} \in (-1,0)$

By using Holder's inequality, and (H_2), according to [31,32], we have

$$\int_0^t |(t-s)^{q-1} g(s, x_s, \int_0^s k(t, x(t))dt)|ds \le \left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}} ds \right)^{1-q_1} N_g N_k$$

$$\le \frac{N_g N_k}{(1+a)^{1-q_1}} b^{(1+a)(1-q_1)} \to (4)$$

From the inequality (4) and Lemma 2.6 [31,32], we obtain the following inequality

$$\int_0^t |(t-s)^{q-1}\zeta(t-s) g(s, x_s, \int_0^s k(t, x(t))dt)|ds \le \frac{Mq}{\Gamma(1+q)} \int_0^t |(t-s)^{q-1} g(s, x_s, \int_0^s k(t, x(t))dt)|ds$$

$$\le \frac{MqN_g N_k}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)} \to (5)$$

According to [32], we obtain the following relation:

$$\int_0^t |(t-s)^{q-1} A\zeta(t-s) f(s, x_s, \int_0^s h(t, x(t))dt)|ds \le \int_0^t |(t-s)^{q-1} A^{1-\beta}A^{\beta}\zeta(t-s) f(s, x_s, \int_0^s h(t, x(t))dt)|ds$$

$$\le \frac{N_{(-A)^\beta f}\Gamma(1+\beta)C_{1-\beta}b^{q\beta}N_h}{\beta\Gamma(1+\beta q)} \to (6)$$

Let x ϵ Y and t ϵ [0,b], we observe from axiom (A) of the phase spaces, we obtain that $\| x_t \|_B \leq K_b \| x \|_b + M_b \| x_0 \|_B \leq r$ this implies that $x_t \in B_r[0,B]$, and this case

$$\| \Gamma x(t) \| \leq \| \Im(t) \| (\| \varphi(0) \| + \| f(0,\varphi) \| + \| q(x_{t_1}, x_{t_2}, \ldots x_{t_n})(0)) \| + \| f(t, x_t, \int_0^t h(s,x(s))ds) \|$$

$$+ \int_0^t (t-s)^{q-1} \| A\zeta(t-s) f(s, x_s, \int_0^s h(t,x(t))dt) \| ds + \int_0^t (t-s)^{q-1} \| \zeta(t-s) g(s, x_s, \int_0^s k(t,x(t))dt) \| ds$$

$$\leq M(H \| \varphi \|_B + N_f + N_q) + N_f N_h + \frac{N_{(-A)^\beta f} \Gamma(1+\beta) C_{1-\beta} b^{q\beta}}{\beta \Gamma(1+\beta q)} + \frac{N_g N_k M q}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)} \to (7)$$

and

$$\| (\Gamma u)_0 \| \leq \| \varphi \| + \| q(x_{t_1}, x_{t_2}, \ldots.. x_{t_n}) \|,$$

$$\leq H \| \varphi \|_B + N_q \to (8)$$

From (7) – (8), we have that

$$\| \Gamma x(t) \|_{S(b)} \leq M_b \| (\Gamma x)_0 \|_B + K_b \| x \|_b$$

$$\leq M_b \left[\| \varphi \|_B + N_q \right] + K_b \left[MH \| \varphi \|_B + MN_f + MN_q + N_f N \right.$$

$$\left. + \frac{K_b N_{(-A)^\beta f} \Gamma(1+\beta) C_{1-\beta} b^{q\beta}}{\beta \Gamma(1+\beta q)} + \frac{K_b N_g N_k M q}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)} \right]$$

$$= \rho < r \to (9)$$

which proves that Γ(x) ϵ Y.

In order to prove that Γ satisfies a Lipschitz condition, u ,v ϵ Y. If t ϵ [0,b], we see that

$$\| \Gamma u(t) - \Gamma v(t) \| \leq \| \Im(t)(q(u_{t_1}, u_{t_2}, \ldots.. u_{t_n})(0) - q(v_{t_1}, v_{t_2}, \ldots.. y_{t_n})(0) \| + \| (-A)^{-\vartheta} \|$$

$$\| (-A)^{\vartheta} f(t, u_t, \int_0^t h(s,u(s))ds) - (-A)^{\vartheta} f(t, v_t, \int_0^t h(s,v(s))ds) \| + \int_0^t (t-s)^{q-1} \| (-A)^{1-\vartheta} \zeta(t-s) \|$$

$$\| (-A)^{\vartheta} f(s, u_s, \int_0^s h(t,u(t))dt) - (-A)^{\vartheta} f(s, v_s, \int_0^s h(t,v(t))dt) \| ds + \int_0^t (t-s)^{q-1} \| \zeta(t-s) \|$$

$$\| g(s, u_s, \int_0^s h(t,u(t))dt) - g(s, v_s, \int_0^s h(t,v(t))dt) \| ds$$

$$\leq M \sum_{i=1}^n L_i(q) \| u_{t_i} - v_{t_i} \|_B + \| (-A)^{-\vartheta} \| L_f \left[\| u_t - v_t \|_B + L_h \| u(s) - v(s) \|_B \right] + L_f [\| u_s - v_s \|_B$$

$$+ L_h \| u(s) - v(s) \|_B] \frac{\Gamma(1+\beta) C_{1-\beta} b^{q\beta}}{\beta \Gamma(1+\beta q)} + L_g \left[\| u_s - v_s \|_B + L_h \| u(t) - v(t) \|_B \right] \frac{Mq}{\Gamma(1+q)(1+a)^{1-q_1}} b^{(1+a)(1-q_1)}$$

$$\leq M_b\left[M\sum_{i=1}^{n}L_i(q)+\|(-A)^{-\vartheta}\|L_f\right]\|u_0-v_0\|_B+K_b\left[M\sum_{i=1}^{n}L_i(q)+L_f\|(-A)^{-\vartheta}\|\right]\|u-v\|_b$$

$$+M_b\left[L_f\frac{\Gamma(1+\beta)C_{1-\beta}b^{q\beta}}{\beta\Gamma(1+\beta q)}+L_g\frac{Mq}{\Gamma(1+q)(1+a)^{1-q_1}}b^{(1+a)(1-q_1)}\right]\|u_0-v_0\|_B$$

$$+K_b\left[L_f\frac{\Gamma(1+\beta)C_{1-\beta}b^{q\beta}}{\beta\Gamma(1+\beta q)}+L_g\frac{Mq}{\Gamma(1+q)(1+a)^{1-q_1}}b^{(1+a)(1-q_1)}\right]\|u-v\|_b+\|(-A)^{-\vartheta}\|L_h\|u(s)-v(s)\|_B$$

$$+L_fL_h\|u(s)-v(s)\|_B\frac{\Gamma(1+\beta)C_{1-\beta}b^{q\beta}}{\beta\Gamma(1+\beta q)}+L_gL_h\|u(t)-v(t)\|_B\frac{Mq}{\Gamma(1+q)(1+a)^{1-q_1}}b^{(1+a)(1-q)_1}$$

$$\leq M_b\theta\|u_0-v_0\|_B+K_b\theta\|u-v\|_b+\theta_1\|u(s)-v(s)\|_B$$

On the other hand, a simple calculus prove that

$$\|(\Gamma u)_0-(\Gamma v)_0\|\leq\sum_{i=1}^{n}L_i(q)\left[K_b\|u-v\|_b+M_b\|u_0-v_0\|_B\right]$$

Finally we see that

$$\|(\Gamma u)-(\Gamma v)\|_{S(b)}\leq M_b\|(\Gamma u)_0-(\Gamma v)_0\|_B+K_b\|(\Gamma u)-(\Gamma v)\|_b$$

$$\leq M_b\left[\sum_{i=1}^{n}L_i(q)+K_b\theta\right]\|u_0-v_0\|_B+K_b\left[M_b\sum_{i=1}^{n}L_i(q)+K_b\theta\right]\|u-v\|_b+K_b\theta_1\|u(s)-v(s)\|_B$$

$$\leq\Lambda\|u-v\|_{S(b)}\rightarrow(10)$$

which infer that Γ is a contraction on Y. Clearly, a fixed point of Γ is the unique mild solution of the nonlocal problem (1) – (2). Hence the proof is complete.

IV. EXAMPLE

In this section, we consider an application of our abstract results. We introduce some of the required technical framework. Here, let $X = L^2([0,\pi])$, $B = C_0 \times L^p(g, X)$ is the space introduced in Example 2.5 and $A : D(A) \subset X$ xX is the operator defined by $Ax = x''$, with $D(A) = \{x \in X: x'' \in X, x(0) = x(\pi) = 0\}$.
The operator A is the infinitesimal generator of an analytic semigroup on X.. Then

$$A=-\sum_{i=1}^{\infty}n^2\langle x,e_n\rangle e_n,\ x\in D(A),$$

where $e_n(\xi)=\left(\frac{2}{\pi}\right)^{1/2}\sin(n\xi), 0\leq\xi\leq\pi, n=1,2,\ldots\ldots$ Clearly, A generates a compact semigroupT(t), t> 0 in X and is given by

$$T(t)x=\sum_{i=1}^{\infty}e^{-n^2t}\langle x,e_n\rangle e_n,\text{ for every } x\in X.$$

Consider the following fractional partial differential system

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(u(t,\xi)+\int_{-\infty}^{t}\int_{0}^{\pi}b(t-s,\eta,\xi)u(s,\eta)d\eta ds\right)=\frac{\partial^2}{\partial\xi^2}u(t,\xi)+\int_{-\infty}^{t}a_0(s-t)u(s,\xi)ds,\ (t,\xi)\in I\times[0,\pi]\rightarrow(11)$$

$$u(t,0)=u(t,\pi)=0,\ t\in[0,b],\qquad\rightarrow(12)$$

$\Theta \leq 0, \xi \in [0,\pi]$ →(13) $u(\theta,\xi)=\phi(\theta,\xi)+\sum_{i=0}^{n}L_iu(t_i+\xi),$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is a Caputo fractional partial derivative of order $0 < \alpha < 1$, n is a positive integer, $0 < t_i < a$, L_i, i = 1,2,….,n, are fixed numbers.

In the sequel, we assume that $\varphi(\theta)(\xi) = \Phi(\theta,\xi)$ is a function in B and that the following conditions are verified.

(i) The functions $a_0 : R \to R$ are continuous and $L_g := \left(\int_{-\infty}^{0} \frac{(a_0(s))^2}{g(s)} ds \right)^{1/2} < \infty$

(ii) The functions $b(s,\eta,\xi)$, $\frac{\partial b(s,\eta,\xi)}{\partial \xi}$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$ for all (s,η) and

$$L_f := \max\left\{ \left(\int_0^\pi \int_{-\infty}^0 \int_0^\pi g^{-1}(\theta) \left(\frac{\partial^i}{\partial \xi^i} b(\theta,\eta,\xi) \right)^2 d\eta d\theta d\xi \right)^{1/2} : i = 0,1 \right\} < \infty.$$

Defining the operators f, g : I x B $\to$ X by

$$f(\psi)(\xi) = \int_{-\infty}^0 \int_0^\pi b(s,\eta,\xi)\psi(s,\eta) d\eta ds,$$

$$g(\psi)(\xi) = \int_{-\infty}^0 a_0(s)\psi(s,\xi) ds.$$

Under the above conditions we can represent the system (11) - (13) into the abstract system (1) – (2). Moreover, f, g are bounded linear operators with $\| f(.)\|_{L(B,X)} \le L_f$, $\| g(.)\|_{L(B,X)} \le L_g$. Therefore, (H_1) and (H_2) are fulfill. Therefore all the conditions of Theorem 3.2 are satisfied. The following result is a direct consequence of Theorem 3.2.

Proposition 4.1. For b sufficiently small there exist a mild solutions of (11) – (13).

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