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Existance Theory for First Order Nonlinear Random Dfferential Equartion

R. N. Ingle[1], M. K. Bhosale[2]

[1](*Dept. of Mathematic/SRTMUN University Nanded /BahirjiSmarakMahavidyalaya, Basmath,Hingoli/ Maharashtra. India)*

[2](*Dept. of MathematicsShri. ChhatrapatiShivajiMaharaj College of EngineeringDist. Ahmednagar/Pune University, Maharashtra. India)*

Abstract: *In this paper, the existence of a solution of nonlinear random differential equation of first order is proved under Caratheodory condition by using suitable fixed point theorem.*

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I Introduction

It is known that many of dynamical solution of some natural, physical or biological phenomena of the universe are nonlinear in nature and may depend upon the past history or future prediction of dynamical process and so such dynamical systems are governed by the nonlinear differential equation involving past history of future consideration called the random differential equations.

The ordinary derivatives have been discussed in the literature since long time. Some monographs in theory of ordinary differential equation are due to [1] and [2] etc. The differential equations are mainly of two type, viz.

(i) differential equation with deviating arguments, and

(ii) differential equation of natural type.

The differential equation with deviating arguments includes the differential equation with delay or advanced arguments. The delay differential equations include the past history of the dynamical system whereas advanced argument differential equation involves the situation that the velocity depends upon the future state of the dynamic system. Similarly the natural differential equation possesses the property that the velocity depends upon the derivative of the past state of dynamical systems. The both type of differential equations are the most active areas of researches in the subject of the theory of differential equations.

The study of the random differential equations of both the type is comparatively rare in the literature. It is worthwhile to note that the dynamical system involving random parameter may depend upon the past history of random variable or the past rate of change of the random variables, and so they can be modeled on the random differential equation in a better way. Hence the study of the random differential equation has got importance in random analysis of the dynamical systems of the universal phenomena.

In this paper we shall study a first order equation of ordinary random differential equation via the fixed point theory for different aspects of the random solutions.

Let R denote the real line, R^n , an Euclidean space with norm | . | defined by $|x| = |x_1| + |x_2| + \ldots\ldots + |x_n|$ for $x = (x_1,x_2,\ldots,x_n)\in R^n$. Let J = [0,1] and closed and bounded intervals in R. Let $C(J,R^n)$ denote the space of all continuous R^n-Valued functions on J equipped with supremum norm $\|.\|_c$ given by

$\|x\|_c = sup_{t\in J}|x(t)|$.

Clearly C = C (J, R^n) is a Banach Space with this norm. Let L^1(J ,R) denote the space of Lebesgueintegrable R – valued function on J equipped with norm $\|.\|_{L^1}$defined by

$\|x\|_{L^1} = \int|x(t)|dt$.

Given a measureable space (Ω, A) and given a measurable function $\phi : \Omega \to C$, consider the first order ordinary random differential equation (RDE)

$$\left.\begin{aligned} &x'(t,\omega) = f(x, x(t,\omega),\omega) \text{ a.e. } t\in J \\ &x_0(\omega) = \Phi(\omega) \end{aligned}\right\} \quad (1.1)$$

for all $\omega \in \Omega$, where $f : J \times C \times \Omega \to R^n$ and $x : \Omega \to C$ is measurable for each $t \in J$. By a random solution of the RDE (1.1) we mean measurable function $x: \Omega \to X \cap C(J, R^n)$ that satisfies the equations in (1.1) on J, where x = $A \subset (J, R^n)$ is the space of all absolutely continuous R^n – valued function on J.

II Caratheodory Theory

1.1 Auxiliary Result :-

Let (Ω, A) be a measurable space and X be a Banach Space. Let β_X be the σ – algebra of all Banach subsets of X. A function $x: \Omega \to X$ is called measurable if $B \in \beta_X$, then $x^{-1}(B) = \{ \omega \in \Omega : x(\omega) \in B \} \in A$

A function $T : \Omega \times X \to X$ is called random operator if $T(\omega, x)$ is measurable in ω for each $x \in X$ and we denote it by $T(\omega, x)= T(\omega)x$. A function $\xi : \Omega \to X$ is called a random fixed point and if ξ is measurable then $T(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$.

1.2

1.3 Existence Results:

In this section we prove the main result for RDE(1.1) under suitable condition. Define a norm on C(J, *R*), which is a separable Banach Space.

We need the following special form of the fixed point theorem.

Theorem 2.1:- Let X be a separable Banach space and let $T: \Omega \times X \to X$ be random operator satisfying for each $\omega \in \Omega$,

(i) $T(\omega)$ is completely continuous,Then either

(ii) The equation $\lambda(\omega)T(\omega) x = x$ has a solution for $\lambda(\omega) = 1$ or

(iii) The set $\varepsilon = \{ u \in X : \lambda(\omega)T(\omega)u = u, 0 < \lambda(\omega)<1\}$ is unbounded.

We need the following definition in the sequel.

Definition2.1: A mapping $f : J \times C \times \Omega \to R^n$ is called L^1-Caratheodory for each $\omega \in \Omega$

(i) $t \to f(t, x, \omega)$ is measurable for all $x \in C$ and

(ii) $x \to f(t, x, \omega)$ is continuous almost everywhere $t \in J$. Further a ω-caratheodory function $f(t, x, \omega)$ is called L^1_ω- Caratheodory if

(iii) for every real number $r > 0$, there exists a measurable function $h_r:\Omega \to L^1_\omega(I, R)$ such that $|f(t, x, \omega)| < h_r(t, \omega)$ a.e. $t \in J$

For each $\omega \in \Omega$ and for all $x \in C$ with $\|x\|_c \le r$.

We consider the following set of assumption in sequel :

(A_1) The function $\omega \to f(t, x, \omega)$ is measurable for all $t \in J$ and $x \in C$.

(A_2) The function $f: J \times C \times \Omega \to R^n$ is continuous and satisfies for each $\Omega \in \Omega$

$$|f(t, x, \omega) - f(t, y, \omega)| \le \frac{\|x-y\|_c}{a+\|x-y\|_c} \text{ a.e. } t \in J \text{ for all } x, y \in C,$$

Where a is some positive real number.

(A_3) The function $f(t, x, \omega)$ is L^1_ω –Caratheodory.

(A_4) There exists a continuous and non-decreasing function $\psi : R^+ \to (0, \infty)$ satisfying for each $\omega \in \Omega$and $t \in J$,

$|f(t, x, \omega)| \le \gamma(t, \omega)\, \psi(\|x\|_c)$ a.e. $t \in J$ for all $x \in C$.

Theorem2.2 :- Assume that the hypothesis (A1),(A3),(A4) hold, further if

$$\int_{\|\Phi(\omega)\|_c}^{\infty} \frac{ds}{\Phi(s)} > \|\gamma(\omega)\|c \text{ for all } \omega \in \Omega \quad \text{------(2.1)}$$

then the RDE (1.1) has a random solution on J.

Proof: Now the RDE (1.1) is equivalent to the random integral equation (RIE)

$$x(t, \omega) = \begin{cases} \Phi(0, \omega) + \int_0^t f(s, x(s, \omega), \omega)\,ds, & \text{if } t \in J \\ \Phi(t, \omega), & \text{If } t \in J \end{cases} \quad \text{------(2.2)}$$

Define an operator $T :\Omega \times X \to X$ by

$$T(\omega)x(t, \omega) = \begin{cases} \Phi(0, \omega) + \int_0^t f(s, x(s, \omega), \omega)\,ds, & \text{if } t \in J \\ \Phi(t, \omega), & \text{If } t \in J \end{cases} \quad \text{-----(2.3)}$$

We show that the operator T satisfies all the conditions of theorem 2.1

Step-I : First we show that T is a random operator on $\Omega \times X$
Note that the function
$\omega \to \int_0^t f(s, x(s,\omega), \omega)\, ds$
can express as a limit of the finite sum of measurable functions, so it measurable. By hypothesis, the function $\omega \to \Phi(t, \omega)$ is measurable for all $t \in J$. Again the sum of two measurable function is measurable and the function
$\omega \to \phi(0, \omega) + \int_0^t f(s, x(s,\omega), \omega) ds$
is measurable. As a result the function
$\omega \to T(\omega)x(\omega)$.
Step II: Next we show that T(ω, x) is a continuous in X for all $\omega \in \Omega$. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ as $n \to \infty$.
Dominated convergent theorem

$$\lim_n T(\omega)x_n(t) = \begin{cases} \lim_n\{ \Phi(0, \omega) + \int_0^t f(s, x_n(s,\omega), \omega) ds, & t \in J \\ \Phi(t, \omega), & t \in J \end{cases}$$

$$= \begin{cases} \Phi(0, \omega) + \int_0^t f(s, x(s,\omega), \omega) ds, & \text{if } t \in J \\ \Phi(t, \omega), & t \in J \end{cases}$$

$$= \quad T(\omega)x(t)$$

For all $\omega \in \Omega$ this shows that T(ω, x) is continuous random operator on $\Omega \times X$
Step III: Here we showthat T(ω, x) is a totally bounded random operator on $\Omega \times X$. Let S be bounded set inX . Then there is a constant r >0 such that $\| \gamma(\omega) \| \le r \quad \forall \omega \in \Omega$. First we show that T(ω, s) is a uniformly bounded set in X . Since f(t, x, ω) is L^1_ω - Caratheodory, we have

$|T(\omega)\, x_n(t)| \le \max\{ |\Phi(0, \omega)|, |\Phi(t, \omega)|\} + \int_0^t |f(s, x_s(s+\theta, \omega), \omega)| ds$

$T(\omega)\, x_n(t) \le \| \Phi(\omega)\|_c + \int_0^t |\gamma(s,\omega)\, \psi(r)| ds$

$\le \| \Phi(\omega)\|_c + \| \gamma(\omega) \|_{L^1}\, \psi(r)$ for all $t \in J$.

Taking the suprimum over t in the above inequality yields that

$\| T(\omega, x)\| \le \| \phi(\omega)\|_c + \| \gamma(\omega) \|_{L^1}\, \psi(r)$ for all $x \in S$ -----(2.4)

Hence the set T(s)is uniformly bounded in X.
Step IV: Next we show that $\{ T(\omega)\, x_n : n \in N\}$ is equi –continuous set in X. Let $x \in S$ be any element then for any $t, \tau \in J$, one has

$| T(\omega, x)(t) - T(\omega, x)(\tau)| \le | \int_0^t f(s, x_n(s,\omega), \omega)\, ds - \int_0^\tau f(s, x_n(s,\omega), \omega) ds |$

$\le \int_\tau^t |f(s, x_n(s,\omega), \omega)| ds$

$\le \int_\tau^t |\gamma(s,\omega)\, \psi(r)| ds$

$\le |p(t, \omega) - p(\tau, \omega)|$ ------(2.5)

Where $p(t, \omega) = \int_\tau^t |\gamma(s,\omega)\, \psi(r)| ds$. Since the function $t \to p(t, \omega)$ is continuous on a compact interval J, it is uniformly continuous onJ for each $\omega \in \Omega$. Hence from the inequality (2.5) it follows that
$|T(\omega)\, x_n(t, \omega) - T(\omega)\, x_n(\tau, \omega)| \to 0$ as $t \to \tau$,
for all $x \in S$ and for $\omega \in \Omega$.
Again $t, \tau \in J$, then by definition of T(t, ω),

$| T(\omega, x)(t) - T(\omega, x)(\tau) | = |\phi(t, \omega) - \phi(0, \omega)| + \int_0^t |f(s, x_n(s,\omega), \omega)| ds - \int_0^\tau |f(s, x_n(s,\omega), \omega)| ds$

$\le |\phi(t, \omega) - \phi(0, \omega)| + \int_\tau^t |f(s, x_n(s,\omega), \omega)| ds$

$\le |\phi(t, \omega) - \phi(0, \omega)| + \int_\tau^t |\gamma(s,\omega)\, \psi(r)| ds$

$= |\phi(t, \omega) - \phi(0, \omega)| + |p(t, \omega) - p(0, \omega)|$

Note that when $t \to \tau$, $t \to 0$ as $\tau \to 0$. Therefor from above inequality
$|T(\omega)\, x_n(t, \omega) - T(\omega)\, x_n(\tau, \omega)| \to 0$ as $t \to \tau$ for all $x \in S$.

Thus in all three cases we have for each $\omega \in \Omega$,
$| T(\omega, x)(t) - T(\omega, x)(\tau) | \to 0$ as $t \to \tau$ for all $x \in S$.
Thus the set T(s) is equi- continues in X. Consequently T(s) relatively compact in view of Arzela – Ascolli Theorem.
Now the random operator $T(\omega, x)$ satisfies all the condition of the theorem (2.1). Hence an application of it yields that either the conclusion (ii) or the conclusion (iii) holds. Below we show that the conclusion (iii) is not possible.
Step V: Let $u \in \varepsilon$ be arbitrary, where ε is a set of X given in the conclusion (iii) of Theorem (2.1) then we have for each $\omega \in$, there exist u in X with $\|u\| = R$ satisfying $u(t, \omega) = \lambda^{-1}T(t, \omega)$

$$u(t, \omega) = \begin{cases} \lambda^{-1}\{\phi(0, \omega) + \int_0^t f(s, u_s(\omega), \omega)\, ds\} \text{ if } t \in J \\ \lambda^{-1}\phi(t, \omega), \text{ if } t \in J \end{cases}$$

for some $0 < \lambda < 1$. If $t \in J$, Then we have
$| u(t, \omega) | = |\phi(t, \omega)| \leq \| \phi(\omega)\|_c$, For all $\omega \in \Omega$.
Again if $t \in J$, then
$| u(t, \omega) | \leq | \lambda^{-1}| \, |\phi(0, \omega)| + | \lambda^{-1}| \int_0^t f(s, u_s(\omega), \omega)\, ds$
Put $w(t, \omega) = \max_{s \in [-r,t]} |u(s, \omega)|$, Then $| u(t, \omega) | \leq w(t, \omega)$ for all $t \in J$ and $\omega \in \Omega$. Then there is a $t^* \in [-r, t]$ such that
$w(t^*, \omega) = | u(t^*, \omega) |$ if $t^* \in J$, then
$| u(t^*, \omega) | \leq \| \phi(\omega)\|_c$,
And the result follows. If $t^* \in J$, then we have
$| u(t^*, \omega) | \leq \| \phi(\omega)\|_c + \int_0^t \gamma(s, \omega)\psi(\|u_s(\omega)\|_c)ds$
$\leq \| \phi(\omega)\|_c + \int_0^t \gamma(s, \omega)\psi(w(s, \omega))ds$
Let $m(t, \omega) = \| \phi(\omega)\|_c + \int_0^t \gamma(s, \omega)\psi(w(s, \omega))ds$, $t \in J$.
Then we have $w(t, \omega) \leq m(t,\omega)$, for all $t \in J$ and $\omega \in \Omega$. Differentiating this equation with respect to t yields
$m'(t, \omega) = \gamma(t, \omega)\, \psi(t, w(t,\omega))$,
$m(0, \omega) = \|\phi(\omega)\|_{L}$
This further implies that
$m'(t, \omega) \leq \gamma(t, \omega)\, \psi(t, m(t,\omega))$,
i.e. $\frac{m'(t,\omega)}{\phi(m(t,\omega))} \leq \gamma(t, \omega)$
Integrate from 0 to t yields

$$\int_0^t \frac{m'(s,\omega)}{\phi(m(s,\omega))}\, ds \leq \int_0^t \gamma(s,\omega) ds$$

By change of variable we get

$$\int_{\|\phi(\omega)\|_c}^{m(s,\omega)} \frac{ds}{\psi(s)} \leq \|\gamma(\omega)\|_{L} < \int_{\|\phi(\omega)\|_c}^{\infty} \frac{ds}{\psi(s)}$$

From this inequality it is follows that there is a constant $M(\omega) > 0$ such that
$| u(t, \omega) | \leq w(t, \omega) \leq m(t, \omega) \leq M(\omega)$, for all $t \in J$ and $\omega \in \Omega$.
Thus all condition of theorem 2.1 are satisfies and hence an application of it yields that the operator equation $x(t, \omega) = T(\omega, x)(t)$ has a random solution on J.
Consequently the RDE(1.1) has a random solution on J.
This completes the proof.

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