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Maximal Characterization and Series Function of Hardy-Sobolev spaces with an Application on Manifolds

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Abstract: *Let M be a complete, non-compact Riemannian manifold, provided with a doubling measurableμ. In this peper we compared the maximal Hardy-Sobolev spaces with the Hajłasz Sobolev space on$M$, and we showed that they can be identified under the assumption of a Poincare inequality. The proof was based on a characterization of L_p on metric- measure spaces.*

Key words: *Hardy-Sobolev space, Hajlasz- Sobolev space, metric measure spaces.*

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I. Introduction and Preliminaries

The series function in the Hardy-Sobolev space if its derivatives lie in the real Hardy spaceL_1', means that a maximal series function of the derivatives is integrated. One of the aims of this paper is, how to define the maximal series function of the derivatives off_r.

For a locally integrated series function f_r on M define the gradient in the sense of distributions, implies

$$\sum_{r=1}^{\infty}\langle\nabla f_r, \varphi\rangle := -\sum_{r=1}^{\infty}\int_C f_r \, div\ \varphi \ df\mu \ . \qquad (1)$$

For all smooth vector fields φ of compact support. Here $div\ \varphi$ is the divergence, defined via z^* acting on 1-forms. Following the ideas from the scalar case see [4,17], a natural grand maximal a series functions would be to take, at a point $x_i \in M$,

$$\sup\left|\int_M \sum_r f_r \ di\ v\ \varphi \ d\mu\right|,$$

where the supermom is taken over some family $\mathcal{T}_1(x_i)$ of test vector fields φ. In [3] defined in terms of atomic decomposition, with an $L_{1-\varepsilon}$-Sobolev space defined by Hajłasz (H_1^1) [8], we identified df for $f_r \in H_1^1$ with elements of the particulate L (Hardy space) of differential forms defined in [1] and use the usual maximal function characterization of H_1 (see [4, 15, 12]).

Out of order to do this; we need to extend the notion of divergence to a broader class of test vector fields. Here we defined a maximal a series functions$(\nabla f_r)^+$, where the test vector fields were, in a sense, only Lipschitz continuous. Furthermore, it was explained that for $f_r \in L^1_{\text{loc}}(M), (\nabla f_r)^+ \leq Nf_r$, at every point of M, and therefore a series function f_r in the homogeneous Hajłasz Sobolev space$\dot{H}L_{1-\varepsilon}$, (see [11]) characterized by the condition $Nf_r \in \mathcal{L}$, also satisfies$(\nabla f_r)^+ \in \mathcal{L}$.

There is difficulty getting the converse, namely, introduce that a series function f_r with $(\nabla f_r)^+ \in L_{1-\varepsilon}(M)$ belongs $\dot{H}L_{1-\varepsilon}$, either by controlling Nf_r or via an atomic decomposition. In appointed, when effort to do this, the trouble here of writing a given test a series functions η_r, with$\int \eta_r = 0$, as the divergence of enough smooth vector field of compactness support. In the Euclidean setting, this can be done by a simple well-known construction including reprised integration with respect to the coordinates (see [4, 5]) which preserves the smoothness with no gain. However, adapting such a construction to a manifold with constants which are independent of the local coordinates is not evident. In addition, if one wants to have a gain of derivatives, the case of$\tau - 0$, which corresponds to starting with $\eta_r \in L^1$ and obtaining a vector field whose components have bounded derivatives, is not possible ([18]).

In Part 2, we define a new Hardy-Sobolev maximal a series functions $(\nabla f_r)^+$, which coincides with that $(\nabla f_r)^+$ used in [2] to define Hardy-Sobolev spaces on Lipschitz domains in$\mathbb{R}^n$, and use it to define the homogeneous maximal Hardy-Sobolev space $\dot{H}L^1_{1-\varepsilon,max}$. In Part 3, we compare this space with the homogeneous

Hajłasz Sobolev space$\dot{H}L_p$. We showed main result, Theorem (3.4), the proof of that, based on Proposition (4.1) , is contained in part 4.

We work on completeness, non-compactness Riemannian manifold$M$. With the distance a series functions ρ_r and the measure μ (volume) given by the Riemannian $\mathbb{R}^n$ structure, we view (M, ρ_r, μ) as a metric measure space, and use $B(x_i, s)$ to denote the metric ball of radius $s > 0$ centered at $x_i \in M$. Denote by $\langle \cdot,\cdot \rangle_{x_i}$ the Riemannian metric on the tangent space$T_{x_i}M$, let $T^*_{x_i}M$ be the cotangent space at x_i, and d the exterior derivative. For a smooth a series functionsf_r, the gradient ∇f_r can be viewed as the image of the 1-form df_r under the isomorphism between $T^*_{x_i}M$ and$T_{x_i}M$, (see [4, 18]).

A series functions will be called Lipschitz continuous, denoted$f_r \in \mathrm{Lip}(M)$, if there exists $C < \infty$ such that

$$\sum_r \left|f_r(x_i) - f_r\left(x_{(i-1)}\right)\right| \leq \sum_r C\rho_r\left(x_i, x_{(i-1)}\right) \quad \forall x_i, x_{(i-1)} \in M.$$

The smallest such constant C will be denoted by$\|f_r\|_{\mathrm{Lip}}$. By $\mathrm{Lip}_0(M)$ we will mean the space of compactly supported Lipschitz functions.

We will assume the measurable μ on M satisfies the following.

Definition 1.1. Let $C > 0$, for M be a Riemannian manifold, such that for all balls $B(x_i, s), x_i \in M, \sigma > 0$ we have

$$\mu\big(B(x_i, 2\sigma)\big) \leq C\mu\big(B(x_i, \sigma)\big). \tag{2}$$

Notice that if M satisfies (2) then

$$\dim(M) < \infty \ , \ \mu(M) < \infty.$$

Lemma 1.2. (see [3, 4]) Let M be a Riemannian manifold satisfying (2), $\tau = log_1 C_2$,$\vartheta \geq 1$. Then for all$\left(x_i, x_{(i-1)}\right) \in M$,

$$\mu\big(B(x_i, \vartheta R)\big) \leq C\vartheta^\tau \mu\big(B(x_i, R)\big).$$

We show definition concern to Poincare inequality on$M$.

Definition 1.3. (see [4]) Let M a Riemannian manifold admits a Poincare inequality (2) for some $\varepsilon \geq 0$ if there exists a constant $C > 0$ such that, for every ball B so $s > 0$.

$$\sum_r \left(\int_B |f_r - (f_r)_B|^{1-\varepsilon+\delta} d\mu\right)^{1/(1-\varepsilon+\delta)} \leq \sum_r C\sigma \left(\int_B |\nabla f_r|^{1-\varepsilon+\delta} d\mu\right)^{1/(1-\varepsilon+\delta)} \tag{3}$$

Whenever f_r and its distributional gradient ∇f_r are $(1-\varepsilon)$-integrated on B.

II. New definition of maximal Hardy – Sobolev Space

From [4], we define a new Hardy-Sobolev space maximal a series functions. Let us first recall the following definition.

Definition 2 .1. (See [4]). Let $f_r \in L^{'}_{1-\varepsilon,loc}(M)$, we define its great maximal a series functions, that means by $(\nabla f_r)^+$ as pursued:

$$\sum_r (\nabla f_r)^+ \ (x_i) := \sup \left|\int \sum_r f_r \varphi_r \, d\mu\right|, \tag{4}$$

so $\varphi \in \mathrm{Lip}_0(M)$ such that for some ball $B := B(x_i, s)$ includes backing φ,

$$\|\varphi\|_\infty \leq \frac{1}{\mu(B)}, \qquad \|\nabla\varphi\|_\infty \leq \frac{1}{s\mu(B)}, \tag{5}$$

where

$$\|\varphi\|_\infty \leq 1.$$

Now we define the divergence$div\,\psi \in C^\infty(M)$, by given a smooth vector field φ with compactness support, so that

$$\int_M \sum_r \langle \nabla f_r, \varphi \rangle_{x_i} d\mu = -\int_M \sum_r f_r \ div\,\psi \ d\mu,$$

and extend this to a locally integrated a series functions f_r on M, in order to define ∇f_r, in the sense of dividend, wherein (1). If this divisional slope coincides with a measurable vector-field valued a series functions, which we again denote by$\nabla(f_r)$, we can take its length in the Riemannian metric,$|\nabla f_r|_{x_i} := \langle \nabla(f_r)_x, \nabla f_r \rangle_{x_i}$, and compute the semi-norms,

$$\sum_r \|\nabla f_r\|_{1+\delta-\varepsilon} := \sum_r \left(\int_M |\nabla f_r|^{1+\delta-\varepsilon} d\mu\right)^{1/1+\delta-\varepsilon}, \qquad \delta - \varepsilon \geq 0.$$

See quantity to φ andψ, so

$$\sum_r (\nabla(f_r)^+ \ (x_i) \coloneqq \sup \left| \int \sum_r \left(\langle \nabla\varphi, \psi \rangle_{x_i} + \varphi \ div \ \psi \right) d\mu \right|, \tag{6}$$

where$s(B)$, is the radius of the ball$B$, we have

$$\sup \varphi \subset B, \quad \|\varphi\|_\infty \le \frac{1}{\mu(B)}, \ \ \|\nabla\varphi\|_\infty \le \frac{1}{s\mu(B)} \tag{7}$$

Observed to both φ, ψ are smooth, the quantity $\left(\langle \nabla\varphi, \psi \rangle_{x_i} + \varphi \ div \ \psi \right)$, idealizes the divergence of the product $\varphi\psi$. See [4, 15, 12] consider fractional derivatives.

Definition 2.2. Let$g_r \in L^\infty(\chi)$, χ denote to domain in M, and ψ be a vector field in $L^\infty(\chi, \iota M)$ we say that in the distributional sense if there exists $g_r \in L^\infty(\chi)$ and $\mu(\chi) < \infty$, such that

$$\int_\chi \sum_r f_r g_r d\mu = -\int_\chi \sum_r \langle \nabla f_r, g_r \rangle_{x_i} d\mu, \tag{8}$$

for all $f_r \in L^1_{1-\varepsilon,loc}(M)$, with and its distributional gradient ∇f_r integrable onχ.

Remarks 2.3. (i) If M is a completeness non-compactness Riemannian manifold satisfying in (2) then $\mu(M) = \infty$ and$\dot{H}L_{1-\varepsilon} \subset S^1_1$.

(ii**)**$L'_{max}(M) = \{ f_r \in L'_{1-\varepsilon,loc}(M) : (f_r)^+ \in L_{1-\varepsilon}(M) \}$, when we used lebesgue theorem deduce $L'_{max}(M) \subset L_{1-\varepsilon}(M)$. The divergence $div \ \psi \in C^\infty(M)$ so that define,

$$\int_M \sum_r \langle \nabla f_r, \psi \rangle_{x_i} d\mu = -\int_M \sum_r f_r div \ \psi \ d\mu,$$

that

$$\sum_r \|f_r div \ \psi\|_\infty \le \frac{1}{s}.$$

(See [4, 16]).

Corollary 2.4. Once M satisfies see Definition 1.3, $\tau < 0$.

Implies,

$$\sum_r \left(\int_B |f_r(x) - f_r(x)_B|^{1-\epsilon} d\mu \right)^{1/\tau} \le C\sigma \sum_r \left(\int_B |\nabla f_r(x)|^{1-\epsilon} d\mu \right)^{1/\tau}, \qquad \tau < 0 .$$

The maximal a series functions characterization of the Hardy-Sobolev space $L'_{1-\varepsilon}$, shown: a series functions in the Hardy-Sobolev space in order to Euclidean case if its derivatives lie in the real Hardy spaceL'_1, in the sense that a maximal a series functions of the derivatives is integral.

The homogeneous Hardy-Sobolev space $\dot{H}L'_{1-\varepsilon}$ in the Euclidean case includes of all locally integrated a series functions $(f_r)_x$ such that$\nabla(f_r)_x \in L(\mathbb{R})$, some definitions can be displayed for this.

Definition 2.5. Let ϕ be vector fields, $\phi \in \eta(x_i)$ for some ball B,$s(B)$ its radius

$$\dot{H}L^1_{1-\varepsilon,max} \coloneqq \{ f_r \in L^1_{1-\varepsilon,loc} : N(\nabla f_r) \in L \}.$$

So $\dot{H}L^1_{1-\varepsilon,max}$ denote to maximal homogeneous Hardy-Sobolev space, where $N(\nabla(f_r)_x)$ is given by

$$\sum_r N(\nabla(f_r)_x)(\nabla(f_x)_x) \coloneqq \sup \left| \int \sum_r (f_r)_x \mathrm{div} \ \phi d\mu \right|.$$

That is$\phi \in L(B, TM)$,

$$\|\phi\|_\infty \le \frac{1}{\mu(B)}, \qquad \qquad \|N\|_\infty \le \frac{1}{\sigma\mu(B)}.$$

We equip this space with the semi-norm

$$\sum_r \|f_r\|_{\dot{H}L^1_{1-\varepsilon,max}} = \sum_r \|N(\nabla f_r(x))\|_{x_i}.$$

Note that the definition of $N(\nabla f_r)$ coincides with that of the maximal functions series $N^{(x)} f_r$ used in [8], to define Hardy-Sobolev spaces on Lipschitz domains in$\mathbb{R}^n$.

We control the maximal a series functions $(\nabla f_r)^+(x_i)$ and incline of f_r in the Point wise sense. Shown following:

Proposition 2.6. Let $f_r \in L_{1-\varepsilon,max}(M)$ and $(\nabla f_r)^1 \in L_{1-\varepsilon}(M)$ primarily defined by (1), is given by a series functions and gratifies,

$$\sum_r |\nabla f_r|_{x_i} \le C \sum_r (\nabla f_r)^+(x_i) \qquad \mu - a.e. x_i.$$

Consequently,

$$\dot{H}L'_{1-\varepsilon} \subset S_1^1,$$

with

$$\sum_r \|(f_r)_x\|_{S_1^1} \leq C \sum_r \|\nabla(f_r)_x\|_{\dot{H}L'_{1-\varepsilon}}.$$

The non-homogeneous Sobolev space $HL'_{1-\varepsilon}$ is then defined as the space of f_r in $L^{\epsilon}(M,\mu)$ with $\sum_r \|\nabla(f_r)_x\|_{x_i} < \infty$. Similarly, we can define the homogeneous space $\dot{H}$ by taking only $f_r \in L^1_{1-\varepsilon,loc}(M)$ with$\|\nabla(f_r)_x\|_{x_i} < \infty$, and considering the resulting space modulo constants. Show define the new maximal homogeneous Hardy-Sobolev space $\dot{H}L'_{1-\varepsilon,max}$

Definition 2.7. Let Q is constant for supremum that is $\psi \in L^{\infty}(B, \iota M)$ to some$B \coloneqq B(Q,s)$, follows

$$\dot{H}L^1_{1-\varepsilon,max} \coloneqq \{f_r \in L^1_{1-\varepsilon,loc} : \mathcal{M}^+(\nabla f_r) \in L^1\},$$

where $\mathcal{M}^+(\nabla(f_r)_x)$ is given by

$$\sum_r \mathcal{M}^+(\nabla f_r)(x) \coloneqq \sup_{Q \in x_i} \left| \int \sum_r f_r \, div\, \psi \, d\mu \right| Q.$$

We equip this space with the semi-norm

$$\sum_r \|(f_r)_x\|_{\dot{H}L'_{1-\varepsilon,max}} = \sum_r \|\mathcal{M}^+(\nabla(f_r)_x)\|_Q, \quad Q \leq 1.$$

In the introduction we have already notice that $\mathcal{M}^+(\nabla(f_r))$ coincides with that of the maximal series function $M^{(1)}(f_r)$ used in [2,4], to define Hardy-Sobolev spaces on Lipschitz domains in $\mathbb{R}^n$.

III. The maximal Hardy-Sobolev space comparison with Haj □asz Sobolev space

As in the homogeneous case, $\dot{H}L'_{1-\varepsilon} \subset S_1^1$, first we define that on metric measurable space $(X, d_{1-\varepsilon}, m)$:

Definition 3.1. (Hajłasz). Let$\epsilon \geq 0$. The homogeneous Sobolev space $\dot{H}L'_{1-\varepsilon}$ is the set of all a series functions $u^2 \in L^1_{1-\varepsilon,loc}$ such that thereexists a measurable a series functions$\tau \geq 0, \tau \in L^{1-\varepsilon}$, satisfying

$$\sum \left|u^2(x_i) - u^2(x_{(i-1)})\right| \leq d \sum \left(x_i, x_{(i-1)}\right)\left(\tau(x_i) + \tau(x_{(i-1)})\right), \quad \tau - a.e. \tag{9}$$

We equip $\dot{H}L'_{1-\varepsilon}$ with the semi-norm

$$\|u^2\|_{\dot{H}L'_{1-\varepsilon}} = \inf_{\tau \text{ satisfies}(9)} \|\tau\|_{1-\varepsilon}, \ \ \epsilon \leq 0.$$

A non-homogeneous version $\dot{H}^1_{1-\varepsilon} = L' \cap \dot{H}^1_{1-\varepsilon}$ can be defined using the norm$\|u^2\|_{\tau} + \|u^2\|_{\dot{H}^1_{1-\varepsilon}}$. For $\tau > 1$ these spaces can be identified with the usual Sobolev spaces in the Euclidean case , see [8] ,and are part of a more general theory of Sobolev spaces on metric-measurable spaces , (see [9] and [10]).

Hardy-Sobolev spaces on domains in $\mathbb{R}^n$ can be defined see [13]. These Hardy spaces can also be characterized, as was done in [7], via a type of maximal function used by [6].

We define this latter maximal function series, which we call a Sobolev sharp maximal a series functions to the case of one derivative in $\mathcal{L}$.

Definition 3.2. Let $N(f_r)$, that $f_r \in L^1_{1-\varepsilon,loc}$, where $s(B)$ is the radius of the ball B, define Nf_r by

$$\sum_r N(f_r)(x_i) = \sup_{B: x_i \in B} \frac{1}{s(B)} \int_B \sum_r |f_r| - |(f_r)_B| d\mu.$$

The above definition is makes sense in any metric-measurable space.

Theorem 3.3. Let μ is the doubling measurable and m denote metric on a metric space, cf [11].

$$H_1 L_{1-\varepsilon} = \{f_r \in L^1_{1-\varepsilon,loc} : Nf_r \in \mathcal{L}\},$$

with

$$\|f_r\|_{H_1 L_{1-\varepsilon}} \sim \|Nf_r\|_1.$$

As $f_r \in L^1_{1-\varepsilon,loc}$ and $N(f_r) \in \mathcal{L}$ then $(f_r)_x$ satisfies

$$\sum_r \left|f_r(x_i) - f_r(x_{(i-1)})\right| \leq Cm \sum_r \left(x_i, x_{(i-1)}\right)\left(Nf_r(x_i) + Nf_r(x_{(i-1)})\right) \tag{10}$$

We have the following theorem.

Theorem 3.4. For $f_r \in L^1_{1-\varepsilon,loc}$, at every point of M, that

$$\sum_r \mathcal{M}^+(\nabla f_r) \leq \sum_r Nf_r. \tag{11}$$

Therefore

$$H_1 L_{1-\varepsilon} \subset \dot{H}L^1_{1-\varepsilon,max},$$

with

$$\sum_r \|(f_r)_x\|_{\dot{H}L^1_{1-\varepsilon,max}} \leq C \sum_r \|(f_r)_x\|_{\dot{H}L_{1-\varepsilon}},$$

then

$$\mathcal{M}^+(\nabla f_r) \approx N f_r$$

and

$$\dot{H}L^1_{1-\varepsilon,max} = \dot{H}L_{1-\varepsilon}.$$

IV. Proof of Theorem 3.4:

Let $f_r \in L^1_{1-\varepsilon,loc}$ and $x_i \in M$. Take $\psi \in \mathcal{T}_1(x_i)$ as in Definition 2.7, associated to a ball B containingx_i.

$$\int \sum_r f_r div\, \psi\, d\mu = 0.$$

So we can write

$$\left|\int \sum_r f_r \operatorname{div} \psi\, d\mu\right| = \left|\int_B \sum_r (f_r - (f_r)_B) \operatorname{div} \psi\, d\mu\right|$$
$$\leq \frac{1}{s\mu(B)} \int_B \sum_r |f_r - (f_r)_B| d\mu \leq \sum_r N f_r(x_i).$$

Here s is the radius of B.Taking the supremum over all suchψ. We get (11).
We proceed now to the proof of the reverse inequality. For this we will need the following.

Proposition 4.1. Let M is a complete Riemannian manifold satisfying (1) and (2). Let B a ball of M,

$$g_r \in L^\infty_0(B) \coloneqq \left\{ g_r \in L^\infty(B) \colon \int_B \tau d\mu = 0 \right\}.$$

Then there exists $\psi \in L^\infty(B, TM)$ such that div $\psi = g_r$,
Holds in the sense of Definition 2.2 (with$\chi = B$), and $\|\psi\|_\infty \leq Cs\|g_r\|_\infty$.
Where C is the constant appearing in (2) and is independent of B andε. Before proving the proposition, we conclude the proof of Theorem 3.4. Again take $f_r \in L^1_{1-\varepsilon,loc}$, $x_i \in M$ and B a ball of radius s containing x_i. If $g_r \in L^\infty_0(B)$, $\|g_r\|_\infty \leq 1$ and we solve div $\psi = g_r$ with ψ as in Proposition 4.1, then,

$$\tilde{\psi} \coloneqq \frac{\psi}{Cs\mu(B)} \in \mathcal{T}_1(x_i),$$

and

$$\left|\int_B \sum_r f_r\, g_r\, d\mu\right| = \left|\int_B \sum_r f_r \operatorname{div} \psi\, d\mu\right| = C\sigma\mu(B) \left|\int_B \sum_r (f_r)_x \operatorname{div}\left(\tilde{\psi}\right) d\mu\right|,$$

thus

$$\frac{1}{s\mu(B)} \int_B \sum_r |f_r - ((f_r)_x)_B| d\mu$$
$$= \frac{1}{s\mu(B)} \sup_{\tau \in L^\infty_0(B), \|\tau\|_\infty \leq 1} \left|\int_B \sum_r (f_r)_x\, \tau\, d\mu\right|$$
$$\leq C \sup_{\tilde{\psi} \in \mathcal{T}_1(x_i)} \left|\int \sum_r (f_r)_x \operatorname{div}\left(\tilde{\psi}\right) d\mu\right|$$
$$= C \sum_r \mathcal{M}^+(\nabla (f_r)_x)(x_i).$$

Taking the supermom on the left over all balls B containingx_i, we get $N(f_r)_x(x_i) \leq C\mathcal{M}^+(\nabla (f_r)_x)(x_i)$.

Proof of Proposition 4.1. Let B be a ball and $\tau \in L^\infty_0(B)$.Consider

$$h \coloneqq \left\{ \mathcal{H} \in \mathcal{L}(B, TM) \colon \exists\, f_r \in L^1_{1-\varepsilon,loc}(M), \quad \mathcal{H} = \nabla (f_r)_x \text{ on } B \right\}.$$

We view h as a subspace of $\mathcal{L}(B, TM)$ with the norm

$$\|\mathcal{H}\|_{\mathcal{L}(B,TM)} = \int_B |h|_{x_i}\, d\mu.$$

Define a linear functional on h by

$$\Lambda(\mathcal{H}) = -\int_B \sum_r g_r f_r d\mu \quad \text{if } \mathcal{H} = \nabla f_r \in h.$$

Λ Is well defined since $\int_B \tau d\mu = 0$ and is bounded on h thanks to the Poincare inequality (2),

$$\left|\sum \Lambda(\mathcal{H})\right| = \left|\int_B \sum_r g_r (f_r - (f_r)_B) d\mu\right| \leq C\sigma \sum_r \|g_r\|_\infty \int_B \sum_r |\nabla f_r| d\mu = C\sigma \sum_r \|g_r\|_\infty \|\mathcal{H}\|_{\mathcal{L}(B,TM)}.$$

The Hahn-Banach theorem shows that Λ can be extended to a bounded linear functional on $\mathcal{L}(B,TM)$ with norm no greater than $C\sigma\|g_r\|_\infty$. By duality, there exists a vector field $\psi \in \mathcal{L}^\infty(B,TM)$ such that

$$\int_B \sum_r \langle \psi, \nabla f_r \rangle_{x_i} d\mu = \sum_r \Lambda(\nabla f_r) = -\int_B \sum_r g_r f_r d\mu.$$

For all $f_r \in L^1_{1-\varepsilon,loc}(M)$ for which$\nabla f_r \in \mathcal{L}(B,TM)$. By Definition 2.2, this means div $\psi = g_r$ on$B$. Moreover

$$\|\psi\|_\infty \leq C\sigma \sum_r \|g_r\|_\infty.$$

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