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Characterisation and Theorems on Quaternion Hermitian Doubly Stochastic Matrix:

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Abstract :*The concepts of quaternion hermitian doubly stochastic are developed, basic theorems and some results for these matrices and characterization are analyzed with examples.*
Key Words :*doubly stochastic matrix, quaternion hermitian doubly stochastic matrix, unitary quaternion hermitian doubly stochastic matrix*.

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I. Introduction

The concepts of quaternion hermitian doubly stochastic matrix are applied. In this paper, [1, 4, 5, 6] the quaternion hermitian doubly stochastic matrix is developed in quaternion matrices. Denoted by A^T is the transpose of A and A^* is the conjugate transpose of A.

Definition 2.1 [3,2]

A matrix $A \in H^{n\times n}$ is said to be doubly stochastic if A* = A and $\sum_{i=1}^{n} a_{ij} = 1, j = 1, 2, \ldots n$ & $\sum_{i=1}^{n} a_{ij} = 1, i = 1, 2, \ldots n$ and all $|a_{ij}| \geq 0$.

If A is doubly stochastic and also hermitian then it is called a quatemion hermitian doubly stochastic matrix.[QHDSM]

Theorem 2.1

Let A be a square matrix. Then A is quaternion hermitian Doubly stochastic iff A = A*.

Proof:

Let $A = (a_{ij})_{n\times n}$ be quaternion hermitian doubly Stochastic matrix.

Then $a_{ij} = \overline{a_{ji}}$ for all i,j $(i,j)^{th}$ entry of A = $a_{ij} = \overline{a_{ji}}$ =

$(j,i)^{th}$ entry of $(\overline{A})$ = $(j,i)^{th}$ entry of $(\overline{A})^T$ = A* => A=A*.

suppose A=A*. then $(i,j)^{th}$ entry of A= $(i,j)^{th}$ entry of $(\overline{A})^T$

(i.e) aij = $\overline{a_{ji}}$ for all i,j

=> A is quaternion hermitian doubly Stochastic matrix.

EXAMPLE 1.1:

$$A = \begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix}$$

$$A^* = \begin{pmatrix} 1 & 2-i+j & -2+i-j \\ 2+i-j & 3 & -4-i+j \\ -2-i+j & -4+i-j & 7 \end{pmatrix}$$

Theorem 2.2

If A and B are n×n quaternion hermitian doubly Stochastic matrices, then

(i) $\frac{1}{2}(\overline{A+B}) = \frac{1}{2}(\overline{A}+\overline{B})$

(ii) $(\overline{AB}) = \overline{A}\ \overline{B}$.

(iii) (AB)* = B*A*.

(iv) $\frac{1}{2}$ (A+B)* = $\frac{1}{2}$ (A*+B*).

(v) (KA)* = KA*, where K is scalar.are also quaternion hermitian doubly stochastic matrices.

Proof:

(i) Let A = $(a_{ij})_{n\times n}$ and B = $(b_{ij})_{n\times n}$ quaternion hermitian doubly matrices then $\frac{1}{2}$ (A+B) = (c_{ij}) is also n×n quaternion hermitian doubly stochastic matrix where $c_{ij} = a_{ij}+b_{ij}$

$(i,j)^{th}$ entry of $\frac{1}{2}(\overline{A+B}) = \frac{1}{2}\overline{c_{ij}} = \frac{1}{2}(\overline{a_{ij}+b_{ij}}) = \frac{1}{2}(\overline{a_{ij}}+\overline{b_{ij}})$.

$= \frac{1}{2}\left\{ (i,j)^{th} \text{ entry of } \overline{A} + (i,j)^{th} \text{ entry of } \overline{B} \right\}$

⇨ $\frac{1}{2}(\overline{A+B}) = \frac{1}{2}(\overline{A}+\overline{B})$.

(ii) Let A = $(a_{ij})_{n\times n}$ and B = $(b_{ij})_{n\times n}$ quaternion hermitian doubly matrices then AB = (c_{ij}) is an n×n quaternion hermitian doubly Stochastic matrix where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

$(i,j)^{th}$ entry of $(\overline{AB}) = \overline{c_{ij}} = (\overline{a_{i1}+b_{1j}+a_{i2}b_{2j}+...a_{in}b_{nj}})$

$= (\overline{a_{i1}+b_{1j}} + \overline{a_{i2}+b_{2j}} +\ldots+ \overline{a_{in}+b_{nj}})$

$= \sum_{k=1}^{n} \overline{a_{ik}}\ \overline{b_{kj}}$ = $(i,j)^{th}$ entry of $(\overline{A}\ \overline{B}) \Rightarrow \left(\overline{AB}\right) = \left(\overline{A}\overline{B}\right)$

(iii) Let A = $(a_{ij})_{n\times n}$ and B = $(b_{ij})_{n\times n}$ quaternion hermitian doubly matrices then $\frac{1}{2}$ (A+B) is an n×n quaternion hermitian doubly stochastic matrices.

$(i,j)^{th}$ entry of (AB)* = $(j,i)^{th}$ entry of $(\overline{AB})$.=$(j,i)^{th}$ entry of $(\overline{A}\ \overline{B})$ = $(i, j)^{th}$ entry of $[(\overline{B})^T(\overline{A})^T]$ = $(i,j)^{th}$ entry of B*A* $\Rightarrow$ (AB)* = B*A*.

(iv) Let A = $(a_{ij})_{n\times n}$ and B = $(b_{ij})_{n\times n}$ quaternion hermitian doubly matrices then $\frac{1}{2}$ (A+B) is an n×n quaternion hermitian doubly stochastic matrix. Since A* and B* are n×n quaternion hermitian doubly Stochastic matrix.

Thus $\frac{1}{2}$ (A+B)* & $\frac{1}{2}$ (B*A*) are of same type.

$\frac{1}{2}$ $(i,j)^{th}$ entry of (A+B)* = $\frac{1}{2}$ $(j,i)^{th}$ entry of $(\overline{A+B})$ = $\frac{1}{2}$ $(j,i)^{th}$ entry of $(\overline{A}\ \overline{B})$= $\frac{1}{2}$ $(j,i)^{th}$ entry of $[(\overline{A})^T+(\overline{B})^T]$ = $\frac{1}{2}$ $(i,j)^{th}$ entry of (A*+B*).

(v) Let A = $(a_{ij})_{n\times n}$ quaternion hermitian doubly stochastic matrix the $(KA)_{n\times n}$ quaternion hermitian stochastic matrix and hence also $(KA)^T_{n\times n}$ quaternion hermitian stochastic matrix.

Since $(A^*)_{n\times n}$ quaternion hermitian doubly stochastic matrix and also $(KA^*)_{n\times n}$ quaternion hermitian stochastic matrix. Hence (KA)* and (KA*) are of the same type.

Also $(i,j)^{th}$ entry of (KA)* = $(i,j)^{th}$ entry of $(\overline{KA})$ = K $\overline{a_{ji}}$ [K is real, $\overline{K}$ = K] = K$(j,i)^{th}$ entry of $\overline{A}$ = K$(i,j)^{th}$ entry of $(i,j)^{th}$ entry of K($\overline{A}$) => (KA)* = KA*.

Where K is real.

EXAMPLE 1.2:

$$A = \begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 4+i-j & -5-i+j \\ 4-i+j & 2 & -5+i-j \\ -5+i-j & -5-i+j & 11 \end{pmatrix}$$

Theorem 2.3

if A and B are n×n quaternion hermitian doubly Stochastic matrices then

(i) $\frac{1}{2}$ (A+B) is quaternion hermitian doubly Stochastic matrix.

(ii) KA is quaternion hermitian Stochastic matrix, where K is real

(iii) $\frac{1}{2}$ (AB+BA) is not an quaternion hermitian doubly Stochastic matrix.

Proof:

Since A* and B* are n×n quaternion hermitian doubly stochastic matrices then A=A* and B+B*.

(i) $\frac{1}{2}$ (A+B)* $=\frac{1}{2}\left(\overline{A+B}\right)^T = \frac{1}{2}\left(\overline{A}+\overline{B}\right)^T = \frac{1}{2}[\left(\overline{A}\right)^T + \left(\overline{B}\right)^T = \frac{1}{2}$ (A*+B*) $=\frac{1}{2}$ (A+B)=> $\frac{1}{2}$ (A+B) is quaternion hermitian doubly stochastic matrix.

(ii) (KA)* = $(\overline{KA})^T = (\overline{K}\ \overline{A})^T = (K\overline{A})^T$[K is real, $\overline{K}$ =K] = K$(\overline{A})^T$ = KA* = KA, where K is real.

⇨ (KA) is hermitian stochastic matrix, where K is real.

(iii) $\frac{1}{2}$ (AB+BA)* = $\frac{1}{2}$ [(AB)*+(BA)*] = $\frac{1}{2}$ (A*B*+B*A*) = $\frac{1}{2}$ (AB+BA) = $\frac{1}{2}$ (AB+BA)

quaternions does not satisfy commute Property

⇨ $\frac{1}{2}$ (AB+BA) is not an quaternion hermitian doubly Stochastic matrix.

Property 2.1

If $A \in H^{n\times n}$ is quaternion hermitian doubly stochastic matrix the A^n is also quaternion hermitian doubly stochastic matrix for $n \le 2$.

PROOF

$$A = \begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 13 & 14+4i-4j & -26-8i+8j \\ 14-4i+4j & 33 & -46+10i-10j \\ -26+8i-8j & -46-10i+10j & 73 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 117 & 152+37i-37j & -284-81i+81j \\ 152-25i+25j & 356 & -498+103i-103j \\ -284+69i-69j & -490-103i+73j & 793 \end{pmatrix}$$

Property 2.2

Products of any two quaternion hermitian doubly stochastic matrices are also doubly stochastic. matrix but not a quaternion hermitian doubly Stochastic matrix.

PROOF:

$$A = \begin{pmatrix} 1 & 2+i-j+k & -2-i+j-k \\ 2-i+j-k & 3 & -4+i-j+k \\ -2+i-j+k & -4-i+j-k & 7 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2-i-k & -2+i+k \\ 2+i+k & 3 & -4-i-k \\ -2-i-k & -4+i+k & 7 \end{pmatrix}$$

$$AB = \begin{pmatrix} 5 & 18+2i+2j+2k & -22+6i+6j+6k \\ 18+2i+2k & 25 & -42+4i-6j+4k \\ -22+6i+6j+6k & -42-4i+2j+4k & 55 \end{pmatrix}$$

AB is not an quaternion hermitian doubly stochastic matrix.

Hence Products of any two quaternion hermitian doubly stochastic matrices are doubly stochastic matrix but not an quaternion hermitian doubly stochastic matrix.

Property 2.3

quaternion hermitian doubly stochastic matrices are not commutative.

PROOF:

$$A = \begin{pmatrix} 1 & 2+i-j+k & -2-i+j-k \\ 2-i+j-k & 3 & -4+i-j+k \\ -2+i-j+k & -4-i+j-k & 7 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2-i-k & -2+i+k \\ 2+i+k & 3 & -4-i-k \\ -2-i-k & -4+i+k & 7 \end{pmatrix}$$

$$AB = \begin{pmatrix} 5 & 18+2i+2j+2k & -22+6i+6j+6k \\ 18+2i+2k & 25 & -42+4i-6j+4k \\ -22+6i+6j+6k & -42-4i+2j+4k & 55 \end{pmatrix}$$

$$BA = \begin{pmatrix} 5 & 18-2i-2k & 22+6i+6k \\ 18-2i-2j-2k & 25 & -42-4i-2j-4k \\ -22+6i-6j+6k & -42-4i+6j-4k & 65 \end{pmatrix}$$

⇨ AB≠BA => quaternion hermitian doubly stochastic matrices are not commutative.

Property 2.4

If A,B $\in H^{n\times n}$ are quaternion hermitian doubly stochastic matrices. Then A+B = 2C where C is another quaternion hermitian doubly stochastic matrix.

PROOF:

$$A = \begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 4+i-j & -5-i+j \\ 4-i+j & 2 & -5+i-j \\ -5+i-j & -5-i+j & 11 \end{pmatrix}$$

$$A+B = \begin{pmatrix} 3 & 6+2i-2j & -7-2i+2j \\ 6-2i+2j & 5 & -9+2i-2j \\ -7+2i-2j & -9-2i+2j & 18 \end{pmatrix}$$

A+B = 2C

$$2\begin{pmatrix} 3/2 & 3+i-j & -7/2-i+j \\ 3-i+j & 5/2 & -9/2+i-j \\ -7/2+i-j & -9/2-i+j & 9 \end{pmatrix}$$

$$C=\begin{pmatrix} 3/2 & 3+i-j & -7/2-i+j \\ 3-i+j & 5/2 & -9/2+i-j \\ -7/2+i-j & -9/2-i+j & 9 \end{pmatrix}$$

Theorem 2.4

Let A be a quaternion hermitian doubly stochastic matrix, then $\frac{1}{2}$(A*+A), where [(A*)*=A] is quaternion hermitian doubly stochastic matrix.

Proof: $\frac{1}{2}$ [(A+A*)]* = $\frac{1}{2}$ [A*+(A*)*]

= $\frac{1}{2}$ (A*+A)[(A*)*=A]

=> $\frac{1}{2}$ (A+A*) is quaternion hermitian doubly stochastic matrix.

Property 2.5

If $A \in H^{n\times n}$ is quaternion hermitian doubly stochastic matrix then $\frac{1}{2}$ (A+A*)=A.

EXAMPLE 1.3:

$$A=\begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix}$$

$$A^*=\begin{pmatrix} 1 & 2-i+j & -2+i-j \\ 2+i-j & 3 & -4-i+j \\ -2-i+j & -4+i-j & 7 \end{pmatrix}$$

$$A+A^*=\begin{pmatrix} 2 & 4 & -4 \\ 4 & 6 & -8 \\ -4 & -8 & 14 \end{pmatrix}$$

½(A+A*) = 2A/2 = A.

$$=\begin{pmatrix} 2 & 4+2i-2j & -4-2i+2j \\ 4-2i+2j & 6 & -8+2i-2j \\ -4+2i-2j & -8-2i+2j & 14 \end{pmatrix}$$

$$=2\begin{pmatrix} 1 & 2+i-j & -2-i+j \\ 2-i+j & 3 & -4+i-j \\ -2+i-j & -4-i+j & 7 \end{pmatrix}$$

property 2.6

if $A \in H^{n\times n}$ is quaternion hermitian doubly stochastic matrix then (A-A*) is null matrix.

EXAMPLE 1.4:

$$A=\begin{pmatrix} 1 & 2+i-j+k & -2-i+j-k \\ 2-i+j-k & 3 & -4+i-j+k \\ -2+i-j+k & -4-i+j-k & 7 \end{pmatrix} A^*=A$$

$$A^*=\begin{pmatrix} 1 & 2+i+j+k & -2+i-j+k \\ 2+i-j+k & 3 & -4-i+j-k \\ -2-i+j-k & -4+i-j+k & 7 \end{pmatrix}$$

A-A* ia a null matrix.

A-A* = 0

$$A-A^*=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 2.2:[2]

A square matrix A is said to be an unitary quaternion hermitian doubly stochastic matrix if AA* = A*A = I.

Theorem 2.5

A be an unitary quaternion hermitian doubly stochastic matrix then A* is also unitary quaternion hermitian doubly stochastic matrix.

Proof:

Since A is unitary quaternion hermitian doubly stochastic matrix, AA* = A*A = I. therefore (A*)*A*+A*(A*)* => AA* = A*A.AA* = A*A=I =>A* is unitary quaternion hermitian doubly stochastic matrix.

$$\text{Example: } A=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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