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# On the Trivariate Spectral Collocation Method of Solution for Two Dimensional Partial Differential Equations arising in Fluid Mechanics

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***Abstract:*** *In this study we introduce the multi-domain Trivariate spectral collocation method for solving nonlinear parabolic partial differential equations that are defined over large time intervals. The main idea is to reduce the size of the computational domain at each subinterval to ensure that veryaccurateresultsareobtainedwithinshortercomputationaltimewhenthespectralcollocation method is applied. The proposed method is based on applying the quasilinearization technique to simplify the nonlinear partial differential equation (PDE) first. The time domain is decomposed into smaller non-overlapping subintervals. Discretization is then performed on both time and space variables using spectral collocation. The approximate solution of the PDE is obtained by solving the resulting linear matrix system at each subinterval independently. When the solution in the first subinterval has been computed, the continuity condition is used to obtain the initial guess in subsequent subintervals. The solutions at different subintervals are matched together along a common boundary. The examples chosen for numerical experimentation include, the system equations governing Numerical solution of the Transient Free Convection in Magneto-Micropolar Fluid past vertical semi-infinite porous plate with Heat Generation, Mass Transfer and Constant Heat Flux subjected to variable Magnetic Field. To demonstrate the accuracy and the effectiveness of the proposed method, the condition number and the error analysis of the system is presented in graphs and tables.*

**Keywords–***Discretization,multi-domain,nonlinear, Quasilinearization,Spectral Collocation*

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## I. INTRODUCTION

Many practical problems in life result to non-linear partial differential equations. This trend is observed especially in chemistry, physics, biology, mathematics and engineering fields. In order to solve the arising nonlinear PDEs, many assumptions and approximations are made. According to [1]many nonlinear PDEs are not solvable analytically.[2, 3] reiterate that investigation of solutions of such non-linear PDEs hasthenbeenofkeyinteresttomanyresearchersduetotheirpotentialapplicationsandmore effort has been devoted to search for better and more efficient solution methods for these nonlinear models. [4] studied Applications of Radial Basis Function Schemes to Fractional Partial Differential Equations. In their work, it is proposed to show that meshless methods, in particular methods with radial basis functions (RBF), are an alternative to schemes in differences or structured meshes.

TheNumericalSolutionofSingularlyPerturbedNonlinearPartial Differential Equations in Three Space Variables, the Adaptive Explicit Inverse Preconditioning Approach has been analyzed by [5]. A modified explicit preconditioned conjugate gradient method based on explicit inverse preconditioners is introduced for solving complex nonlinear parabolic problems.ThenumericalsolutionofacharacteristicSPnonlinear initial/boundary value is presented, and numerical results demonstrating both applicability and effectiveness of the derived new methods are given.Two-Dimensional Meshless Solution of theNon-Linear Convection-Diffusion-ReactionEquation by the Local HermitianInterpolation Method was studied by [6]. The numericalscheme is verified by comparing the obtained results to the one-dimensionalBurgers' and two-dimensional Richards' analytical solutions. The same resultsare obtained for all the non-linear solvers tested, but better convergence ratesare attained with the Newton Raphson method in a double iteration scheme.

[7] analyzed a novel RBF-FD meshless scheme in curvilineargeometry for unbounded flows. They found out that the order of accuracy ofthe method is found in comparison with a finite difference scheme.A spectral collocation method based on integratedChebyshev polynomials for two-dimensionalbiharmonic boundary-value problems was presented [8]. The performance of the proposed method is investigated

byconsidering several biharmonic problems of first and second kinds; more accurate resultsand higher convergence rates are achieved than with conventional differential methods. [9] presented an Introduction to Spectral/Pseudo spectral Methods.They defined the spectral space representation of functions and the transformation to the physical space representation. A Hilbert space is defined as well as the definition of self-adjoint operators that occur in quantum mechanics and kinetic theory. A time-spectral method for initial-value problemsusing a novel spatial subdomain scheme has been formulated by [10]. They established that the Common Boundary-Condition method (CBC) here proposed is a spatial subdomain scheme for the GWRM. It solves the physical equations independently from the global connection of subdomains in order to reduce the totalnumber of modes. Thus, it is a condensation procedure in the spatial domain thatallows for a simultaneous global temporal solution. It is here evaluated against thefinite difference methods of Crank-Nicolson and Lax-Wendroff for two examplelinear PDEs. The CBC-GWRM is also applied to the linearised ideal Magnetohydrodynamics (MHD) equations for a screw pinch equilibrium. The growth rate of the mostunstable mode was efficiently computed with an error<0:1%.

Inthisstudy,wedescribethemulti-domain Trivariatespectralcollocationmethod(MDTSCM) to solutions of nonlinear parabolic PDEs defined over large time interval. The MDTSCM is based on decomposing the given domain of approximation in the time variable into smaller subintervals and then solving the PDE independently in each subinterval using the Trivariate spectral collocation method. The multi-domain approach has been applied to solve nonlinear ordinary differential equations that model chaotic systems described as 1st order systems of equations, [11]-[13]. In this study the same idea is extended to solutions of non-linear parabolic PDEs. In the description of the method, the algorithm is kept as simple as possible, whileretainingtheheartofgeneralitytocovermanyapplications. Theextentofthediscussion of multi-domain approach in this study is limited to non-overlapping subintervals only.

## II. METHOD OF SOLUTION

In this section we describe the algorithm to describe how the multi-domain Trivariate spectral collocation method can be applied to solve nonlinear parabolic PDEs. We shall consider a general second order nonlinear system of PDEs arising from the problem of Numerical solution of the Transient Free Convection in Magneto-Micropolar Fluid past vertical semi-infinite porous plate with Heat Generation, Mass Transfer and Constant Heat Flux subjected to variable Magnetic Field as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-S\frac{\partial u}{\partial z}=(1+\Delta)\frac{\partial^2 u}{\partial z^2}+Gr\theta+Gc\phi+\Delta\frac{\partial N}{\partial z}-\left(M+\frac{1}{K}\right)u \tag{1}$$

$$\frac{\partial N}{\partial t}+u\frac{\partial N}{\partial x}-S\frac{\partial N}{\partial z}=\lambda\frac{\partial^2 N}{\partial z^2}-\frac{\Delta}{j}\left(2N+\frac{\partial u}{\partial z}\right) \tag{2}$$

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}-S\frac{\partial \theta}{\partial z}=\frac{1}{\Pr}\frac{\partial^2 \theta}{\partial z^2}+\alpha\theta+Ec(1+\Delta)\left(\frac{\partial u}{\partial z}\right)^2 \tag{3}$$

$$\frac{\partial \phi}{\partial t}+u\frac{\partial \phi}{\partial x}-S\frac{\partial \phi}{\partial z}=\frac{1}{Sc}\frac{\partial^2 \phi}{\partial z^2} \tag{4}$$

subject to boundary conditions

$$\begin{aligned} &u(x,0,t)=0,\ N(x,0,t)=0,\ \theta(x,0,t)=1,\ \phi(x,0,t)=1\\ &u(x,\infty,t)=0,\ N(x,\infty,t)=0,\ \theta(x,\infty,t)=1,\ \phi(x,\infty,t)=1 \end{aligned} \tag{5}$$

and the initial conditions

$$u(x,y,0)=0,\ N(x,y,0)=0,\ \theta(x,y,0)=0,\ \phi(x,y,0)=0 \tag{6}$$

In view of the non-linearity in Eq.(3) and the coupling of the partial diff erential equations Eqs.1-4, we simplify the diff erential equations using relaxation method. In the relaxation method, it is assumed that all nonlinear terms are known from the previous iteration while at the same time decoupling the system of diff erential equations using the Gauss-Seidel approach. Applying the relation method, we obtained the decoupled system of equations;

$$(1+\Delta)\frac{\partial^2 u_{s+1}}{\partial z^2}+S\frac{\partial u_{s+1}}{\partial z}-\left(M+\frac{1}{K}\right)u_{s+1}-\frac{\partial u_{s+1}}{\partial t}=-u_s\frac{\partial u_s}{\partial x}-Gr\theta_s-Gc\phi_s-\Delta\frac{\partial N_s}{\partial z} \tag{7}$$

$$\lambda\frac{\partial^2 N_{s+1}}{\partial z^2}+S\frac{\partial N_{s+1}}{\partial z}-u_{s+1}\frac{\partial N_{s+1}}{\partial x}+\frac{2\Delta}{j}N_{s+1}-\frac{\partial N_{s+1}}{\partial t}=-\frac{\Delta}{j}\frac{\partial u_{s+1}}{\partial z} \tag{8}$$

$$\frac{1}{\Pr}\frac{\partial^2 \theta_{s+1}}{\partial z^2}+S\frac{\partial \theta_{s+1}}{\partial z}-u_{s+1}\frac{\partial \theta_{s+1}}{\partial x}-\frac{\partial \theta_{s+1}}{\partial t}-\alpha\theta_{s+1}=-Ec(1+\Delta)\left(\frac{\partial u_{s+1}}{\partial z}\right)^2 \tag{9}$$

$$\frac{1}{Sc}\frac{\partial^2\phi_{s+1}}{\partial z^2}+S\frac{\partial\phi_{s+1}}{\partial z}-u_{s+1}\frac{\partial\phi_{s+1}}{\partial x}-\frac{\partial\phi_{s+1}}{\partial t}=0 \tag{10}$$

where small $s$ signifies the previous iteration level. Using initial approximations to solutions of the partial differential equations as $u_0$, $N_0$, $\theta_0$, and $\phi_0$, the Gauss-Seidel relaxation scheme Eqs.(7)-(10) is solved iteratively until the solution converges. As a rule of thumb, a simple choice of the initial approximation to the solution is a polynomial that satisfies the given boundary conditions. The semi-infinite domain of approximation $[0,\infty)$ is truncated into a finite domain $[0,L]$ where $L$ is taken to be large enough to approximate conditions at infinity. The finite domain of approximation is discretized into Chebyshev Gauss-Lobatto nodes defined in Eq. (11) as

$$\begin{aligned}
\{\hat{x}_i\}_{i=0}^{N_x} &= \cos\left(\frac{i\pi}{N_x}\right),\ \hat{x}(x)=\frac{2}{b-a}\left[x-\frac{1}{2}(b+a)\right],\ x\in[a,b],\\
\{\hat{z}_j\}_{j=0}^{N_z} &= \cos\left(\frac{j\pi}{N_z}\right),\ \hat{z}(z)=\frac{2}{L}\left[z-\frac{L}{2}\right],\ z\in[0,L],\\
\{\hat{t}_k\}_{k=0}^{N_t} &= \cos\left(\frac{k\pi}{N_t}\right),\ \hat{t}(t)=\frac{2}{T}\left[t-\frac{T}{2}\right],\ t\in[0,T]
\end{aligned} \tag{11}$$

Fig1 depicts the Grid points/ collocation points used to estimate the solution.

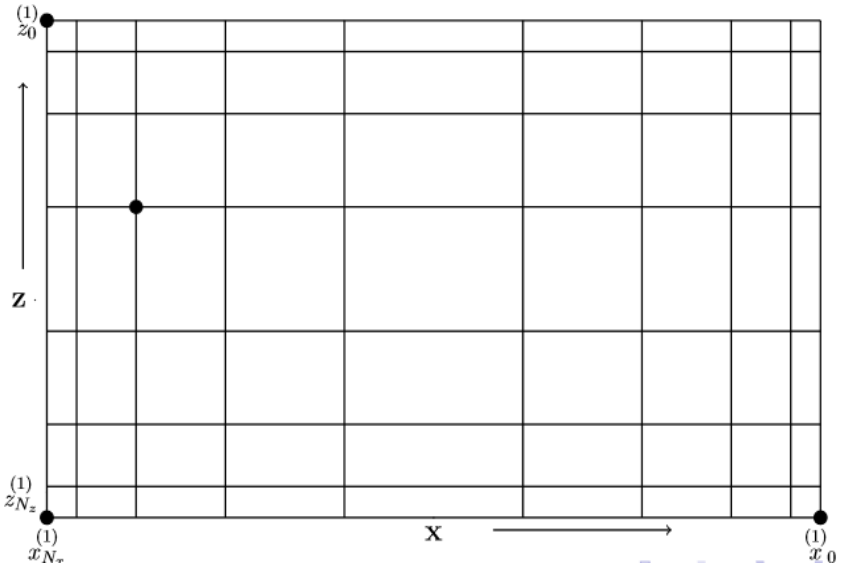


**Figure 1:** Structure of the grid in each spatial domain

Here $[a,b]$ is the interval of approximation in the $x$-direction and $T$ is the final time. The objects $N_x$, $N_z$, $N_t$ are the grid points in $x$, $z$, and $t$, respectively. In the solution process, the approximate solution of the PDEs Eqs. (1)-(6) is assumed to be the Trivariate Lagrange interpolating polynomials. For illustrative purposes, we shall consider the solution to the unknown function $u(x,z,t)$ that takes the form;

$$u(x,z,t)\approx U(x,z,t)=\sum_{p=0}^{N_x}\sum_{q=0}^{N_z}\sum_{r=0}^{N_t}U(x_p,z_q,t_r)L_p(x)L_q(z)L_r(t) \tag{12}$$

The spatial differentiation matrix in $x$ is approximated at the collocation nodes $(\hat{x}_i,\hat{y}_j,t_k)$, for $j=0,1,2,\ldots\ldots,N_z$, and $k=0,1,2,\ldots\ldots,N_t$, as follows;

$$\frac{\partial u}{\partial t}(\hat{x}_i,\hat{z}_j,\hat{t}_k)\approx\sum_{p=0}^{N_x}\sum_{q=0}^{N_z}\sum_{r=0}^{N_t}U(x_p,z_q,t_r)L'_p(\hat{x}_i)L_q(\hat{z}_j)L_r(\hat{t}_k)=\sum_{p=0}^{N_x}U(x_p,z_q,t_k)L'_p(\hat{x}_i)=D\bigcup_k^j=\left(\frac{2}{b-a}\right)D\bigcup_k^j \tag{13}$$

Where $\hat{D} = \left(\frac{b-a}{2}\right)D$ is the standard first order Chebyshev differentiation matrix of size $(N_x+1)\times(N_x+1)$ as defined in [1]. The higher order differentiation matrices are obtained using matrix multiplication. The vector $\bigcup_k^j$ is defined as

$$\bigcup_k^j = \left[u(x_0, z_j, t_k), u(x_1, z_j, t_k), \ldots\ldots\ldots, u(x_{N_x}, z_j, t_k)\right],\ j = 0,1,\ldots,N_z,\ k = 0,1,\ldots,N_t \tag{14}$$

where $T$ denotes matrix transpose. Similarly, the spatial differentiation matrix in **y** is approximated at the collocation points $(\hat{x}_i, \hat{z}_j, \hat{t}_k)$, for $i = 0,1,2\ldots,N_x$ and $k = 0,1,2,\ldots,N_t$ as

$$\frac{\partial u}{\partial z}(\hat{x}_i, \hat{y}_j, \hat{t}_k) \approx \sum_{q=0}^{N_z} U(x_i, y_q, t_k) L'_q(\hat{z}_j) = \sum_{q=0}^{N_z} D_{j,q} \bigcup_k^q = \sum_{q=0}^{N_z} \left(\frac{2}{L}\right) \hat{D}_{j,q} \bigcup_k^q \tag{15}$$

Where $\hat{\hat{D}} = \left(\frac{L}{2}\right)\overline{D}_{j,q}$, $j,q = 0,1,2,\ldots\ldots,N_z$, are entries of a standard first order Chebyshev differentiation matrix of size $(N_z+1)\times(N_z+1)$. Higher order differentiation matrix with respect to $z$ can be obtained using matrix multiplication. Finally, we approximate the differentiation matrix in $t$ at the collocation points $(\hat{x}_i, \hat{z}_j, \hat{t}_k)$, for $i = 0,1,2\ldots,N_x$, and $j = 0,1,2,\ldots,N_z$, as;

$$\frac{\partial u}{\partial t}(\hat{x}_i, \hat{z}_j, \hat{t}_k) \approx \sum_{r=0}^{N_t}\sum_{q=0}^{N_z} U(x_i, y_j, t_r) L'_r(\hat{t}_k) = \sum_{q=0}^{N_z} D_{j,q} \bigcup_k^q = \sum_{r=0}^{N_t}\sum_{q=0}^{N_z} \overline{\overline{D}}_{k,r} \bigcup_r^q \tag{16}$$

Where $\hat{\hat{\hat{D}}}_{k,r} = \left(\frac{T}{2}\right)\overline{\overline{D}}_{k,r}$, $k,r = 0,1,2,\ldots\ldots,N_t$, are entries of a standard first order Chebyshev differentiation matrix of size $(N_t+1)\times(N_t+1)$. We remark that the bar in $\overline{D}$ at Eq. (15) and double bar in $\overline{\overline{D}}$ at Eq. (16) distinguishes the differentiation matrix in $z$ and $t$, respectively, from that in $x$. We note that in generating the sequence of vectors $\bigcup_k^j$, $j = 0,1,2,\ldots\ldots,N_z$, $k = 0,1,2,\ldots\ldots,N_t$, the superscript $j$ is varied of each subscript $k$. Such a pattern will be useful when assembling the system of linear algebraic equations to obtain coefficient matrices. The partial derivatives of the unknowns $N$, $\theta$, $\phi$ are approximated in a similar manner.

Using Eq. (13), Eq. (15) and Eq. (16) in the scheme Eqs. (1)-(6), we obtain a $(N_t+1)\times(N_y+1)\times(N_x+1)$ decoupled system of linear algebraic equations given by;

$$\left(M + \frac{1}{K}\right)\bigcup_k^j + \sum_{r=0}^{N_t}\left[(1+\Delta)\overline{D}_{j,q}^2 + S\overline{D}_{j,q}\right]\bigcup_k^q - \sum_{r=0}^{N_t}\sum_{q=0}^{N_z}\overline{\overline{D}}_{k,r}\bigcup_r^q = R_1 \tag{17}$$

$$\left[-u_{s+1}D + \frac{2\Delta}{j}\mathrm{I}\right]\mathrm{N}_r^q + \sum_{q=0}^{N_z}\left[\lambda\overline{D}_{j,q}^2 + S\overline{D}_{j.q}\right]\mathrm{N}_k^q - \sum_{r=0}^{N_t}\sum_{q=0}^{N_z}\overline{\overline{D}}_{k,r}\mathrm{N}_r^q = R_2 \tag{18}$$

$$\left[-u_{s+1}D\right]\Theta_r^q + \sum_{q=0}^{N_z}\left[\frac{1}{\mathrm{Pr}}\overline{D}_{j,q}^2 + S\overline{D}_{j.q}\right]\Theta_k^q - \sum_{r=0}^{N_t}\sum_{q=0}^{N_z}\overline{\overline{D}}_{k,r}\Theta_r^q = R_3 \tag{19}$$

$$\left[-u_{s+1}D\right]\Phi_k^j + \sum_{q=0}^{N_z}\left[\frac{1}{Sc}\overline{D}_{j,q}^2 + S\overline{D}_{j.q}\right]\Phi_k^q - \sum_{r=0}^{N_t}\sum_{q=0}^{N_z}\overline{\overline{D}}_{k,r}\Phi_r^q = R_4 \tag{20}$$

where I is an identity matrix of size $(N_x+1)\times(N_x+1)$. The right-hand side of equations Eqs. (17)-(20) is defined as

$$\left.\begin{aligned} R_1 &= -u_s\frac{\partial u_s}{\partial x} - Gr\,\theta_s - Gc\,\phi_s - \Delta\frac{\partial \mathrm{N}_s}{\partial z},\ R_2 = -\frac{\Delta}{j}\frac{\partial u_{s+1}}{\partial z}, \\ R_3 &= -Ec(1+\Delta)\frac{\Delta}{j}\left(\frac{\partial u_{s+1}}{\partial z}\right)^2,\ R_4 = 0 \end{aligned}\right\} \tag{21}$$

where**0** is a zero vector of size $(N_t+1)\times(N_y+1)\times(N_x+1)\times 1$. The initial condition $u(x,z,t)$ when evaluated at the collocation points yields

$$u(x_i,z_j,0) = u(x_i,y_j,t_{N_t}) = 0 = \underset{N_t}{\overset{j}{\boldsymbol{U}}} \tag{22}$$

The initial conditions corresponding to the other unknowns are evaluated in a similar manner. The initial conditions reduce Eqs. (17)-(20) to

$$\left(M+\frac{1}{K}\right)\bigcup_k^j + \sum_{q=0}^{N_z}\left[(1+\Delta)\overline{D}_{j,q}^2 + S\overline{D}_{j,q}\right]\bigcup_k^q - \sum_{r=0}^{N_t-1}\sum_{q=0}^{N_z}\overline{\overline{D}}_{k,r}\bigcup_r^q = R_1 \tag{23}$$

$$\left[-u_{s+1}D + \frac{2\Delta}{j}\mathrm{I}\right]\mathrm{N}_k^j + \sum_{q=0}^{N_z}\left[\lambda\overline{D}_{j,q}^2 + S\overline{D}_{j.q}\right]\mathrm{N}_k^q - \sum_{r=0}^{N_t-1}\sum_{q=0}^{N_z}\overline{\overline{D}}_{k,r}\mathrm{N}_r^q = R_2 \tag{24}$$

$$\left[-u_{s+1}D\right]\Theta_k^j + \sum_{q=0}^{N_z}\left[\frac{1}{\mathrm{Pr}}\overline{D}_{j,q}^2 + S\overline{D}_{j.q}\right]\Theta_k^q - \sum_{r=0}^{N_t-1}\sum_{q=0}^{N_z}\overline{\overline{D}}_{k,r}\Theta_r^q = R_3 \tag{25}$$

$$\left[-u_{s+1}D\right]\Phi_k^j + \sum_{q=0}^{N_z}\left[\frac{1}{Sc}\overline{D}_{j,q}^2 + S\overline{D}_{j.q}\right]\Phi_k^q - \sum_{r=0}^{N_t-1}\sum_{q=0}^{N_z}\overline{\overline{D}}_{k,r}\Phi_r^q = R_4 \tag{26}$$

$$\left.\begin{aligned} &u_{s+1}(x,zN_z,t)=0,\ N_{s+1}(x,zN_z,t)=0,\ \theta_{s+1}(x,zN_z,t)=1,\ \phi_{s+1}(x,zN_z,t)=1 \\ &u_{s+1}(x,z_0,t)=0,\ N_{s+1}(x,z_0,t)=0,\ \theta_{s+1}(x,z_0,t)=1,\ \phi_{s+1}(x,z_0,t)=1 \end{aligned}\right\} \tag{27}$$

Eqs. (23)-(26) are solved subject to the boundary conditions

$$\left.\begin{aligned} &u_{s+1}(x,zN_z,t)=0,\ N_{s+1}(x,zN_z,t)=0,\ \theta_{s+1}(x,zN_z,t)=1,\ \phi_{s+1}(x,zN_z,t)=1 \\ &u_{s+1}(x,z_0,t)=0,\ N_{s+1}(x,z_0,t)=0,\ \theta_{s+1}(x,z_0,t)=1,\ \phi_{s+1}(x,z_0,t)=1 \end{aligned}\right\} \tag{28}$$

to yield the approximate numerical solution.

### 2.1 Convergence and stability of the TSCM

#### 2.1.1 Convergence

For a numerical scheme to be useful it needs to correspond to the partial differential equation that is approximated. A numerical method is said to be convergent if the solution of the discrete equations tends to exact solution of the differential equation as the distance between the computational grids is defined. The significance of spectral methods is that they can achieve high accuracy with little more resolution than is required to achieve moderate accuracy [14]. The fundamental problem of the numerical analysis in the boundary value problems is to find the approximate solution $u(x,z,t)$ which converges to exact solution as $N_z$ increases for some given time interval $[0,T]$. To estimate the error, the estimated solution is subtracted from the exact solution. The primary result is the Lax-Richtmyer equivalence theorem which states that stability is equivalent to convergence for consistent approximations to well-posed linear problems[14]. It is important to note that the theorem is applicable to any discretisation; real fluid dynamics is usually nonlinear and a typical problem is usually boundary value or mixed initial and boundary value problems. Let the infinity norm error is approximated as

$$E_i = \left\| u_i^n - u_i^* \right\|_\infty,\ 0 \le i \le N_z \tag{29}$$

where $u_i^n$ is the approximated solution, $u_i^*$ is the exact solution at time level $(t)$ and $N_z$ represent collocation points in the space direction. The scheme is consistent if as $N_z$ tends to infinity, infinity norm error goes to zero and then the scheme is said to be convergent [15]. For nonlinear evolution problems which are influenced by boundary conditions, convergence and stability are difficult to prove. Convergence can be proved by repeating the experiments many times.

In order to demonstrate the convergence of the iterative scheme, we keep track of the infinity error norm of the solutions between successive iterations. We shall call these values the solution error. The solution error at each iteration is calculated using the formula;

$$Sol_e[u] = \|U_{s+1} - U_s\|_\infty,\ Sol_e[N] = \|N_{s+1} - N_s\|_\infty,\ Sol_e[\theta] = \|\Theta_{s+1} - \Theta_s\|_\infty,\ Sol_e[\phi] = \|\Phi_{s+1} - \Phi_s\|_\infty \quad (30)$$

### 2.1.2 Stability

Since the Trivariate Spectral Collocation Method will to be used, we wish to know the largest time-step consistent with stability. The stability analysis for simultaneous partial differential equations is outlined in the MatLab Software. Stability of the iterative numerical scheme is accessed by computing the conditional numbers of the coefficient matrix of the system of linear algebraic equations being solved. To compute the condition number of coefficient matrix **A**, we invoke the MatLab cond(**A**).

## 2.2 Accuracy and Consistency of the MDTSCM

Numerical schemes are used to approximate the solution of evolution equations. In the algorithm development, when approximating the function $u(x,z,t)$ there are errors that arise. The errors that are associated with the algorithm development of the
Numerical scheme are:

- ***Convergence error:*** it is the difference in numerical solution and exact solutions of the given equations. It is also called iteration error.
- ***Modelling error:*** Modelling errors arise due to the difference between the real problem and its formulation as a mathematical model.
- ***Truncation error:*** Truncation error refers to the error in a method, which occurs because some series (finite or infinite) is truncated to a fewer number of terms. Such errors are essentially algorithmic errors and can predict the extent of the error that will occur in the method.
- ***Round off error:*** Round off error occur because of the computing machine inability to deal with certain numbers. Such numbers need to be rounded off to some near approximation which is dependent on the word size used to represent numbers of the device.

A scheme is consistent if the operator reduces to the original differential equation as the increases in the independent variables vanish. Consistency requires that the original equations can be recovered from the arithmetical equations. Clearly this should be a minimum requirement for any discretisation. Consistency is necessary for convergence, but not every consistent scheme is convergent [16]. To achieve the usefulness of the spectral method it is crucial to design it to give greater accuracy than can be obtained using other methods like finite difference methods. The choice of spectral method representation depends on the kind of boundary conditions involved in the problem.

To assess the accuracy of the numerical scheme, we compute the magnitude of the residual error. In order to define the residual error, it is convenient to rewrite Eqs. (3)-(6) in the form;

$$F_v(u, N, \theta, \phi) = 0, \quad v = 1,2,3,4 \quad (31)$$

where $F_v = 0$, $v = 1,2,3,4$ is a nonlinear operator acting on entities in the $v-th$ equation. Residual error in each partial differential equation is thus defined as

$$\mathrm{Res}[F_v] = \|F_v(U, \mathrm{N}, \Theta, \Phi)\|_\infty, \quad v = 1,2,3,4 \quad (32)$$

where $U$, N, $\Theta$, and $\Phi$ are vectors representing discrete solutions for the unknowns $u$, $N$, $\theta$, and $\phi$, respectively.

In the next section, we describe error bound theorems emanating from Trivariate Lagrange interpolating polynomials.

## 2.3 Error bounds theorems in a Multi-Domain Trivariate polynomial interpolation

In this section, we investigate an upper bound for the absolute error. Also, we present a procedure to estimate the absolute error. In this section, we present new error bound theorems that govern polynomial interpolation error in a Trivariate Lagrange interpolating polynomial constructed using Chebyshev Gauss-Lobatto nodes which are essentially the relative extremes of the $N_x - \mathrm{th}$ degree Chebyshev polynomial of the first kind $T_{N_x}(\hat{x}) = \cos[N_x \arccos(\hat{x})],\ \hat{x} \in [-1,1]$. A complete set of the Chebyshev Gauss Lobatto nodes are the roots of the $N_x + 1 - \mathrm{th}$ degree polynomial given by;

$$L_{N_x+1}(\hat{x}) = (1-\hat{x}^2)T'_{N_x}(\hat{x}) \quad (33)$$

The theorem that benchmarks formulation of the error bound theorems on Trivariate polynomial interpolation is given below;

**Theorem 1.** Let $u(x,z,t)\in C^{N_x+N_z+N_t+3}([a,b]\times[c,d]\times[0,T])$ be sufficiently smooth such that at least the $(N_x+1)-$th partial derivative with respect to $x$, $(N_z+1)-$th partial derivative with respect to $z$, $(N_t+1)-$th partial derivative with respect to $t$, and the $(N_x+N_z+N_t+1)-$th mixed partial derivative with respect to $x,z$ and $t$ exists and are all continuous, then there exists values $\varepsilon_x,\varepsilon_x^{'}\in(a,b)$, $\varepsilon_z,\varepsilon_z^{'}\in(c,d)$, and $\varepsilon_t,\varepsilon_t^{'}\in(0,T)$, such that

$$\begin{aligned}u(x,z,t)-U(x,z,t)=&\frac{\partial^{N_x+1}u(\varepsilon_x,z,t)}{\partial x^{N_x+1}(N_x+1)!}\prod_{i=0}^{N_x}(x-x_i)+\frac{\partial^{N_z+1}u(x,\varepsilon_z,t)}{\partial z^{N_z+1}(N_z+1)!}\prod_{j=0}^{N_z}(z-z_j)+\frac{\partial^{N_t+1}u(x,z,\varepsilon_t)}{\partial t^{N_t+1}(N_t+1)!}\prod_{k=0}^{N_t}(t-t_k)\\&-\frac{\partial^{N_x+N_z+N_t+3}u(\varepsilon_x^{'},\varepsilon_z^{'},\varepsilon_t^{'})}{\partial x^{N_x+1}\partial z^{N_z+1}\partial t^{N_t+1}(N_x+1)!(N_z+1)!(N_t+1)!}\prod_{i=0}^{N_x}(x-x_i)\prod_{j=0}^{N_z}(z-z_j)\prod_{k=0}^{N_t}(t-t_k)\end{aligned}\tag{34}$$

where $U(x,z,t)$ is a Trivariate interpolating polynomial of $u(x,z,t)$ at $\{x_i\}_{i=0}^{N_x}$ grid points in $x$-variable, $\{z_j\}_{j=0}^{N_z}$ grid points in $z$-variable, and $\{t_k\}_{k=0}^{N_t}$ grid points in $t$-variable. The remainder formula Eq. (34) is based on the mean value theorem and is derived recursively from to the corresponding univariate error formula given in Eq. (42) for a sufficiently smooth function $u(x,z,t)$. Taking the absolute value of Eq. (34) we obtain

$$\begin{aligned}|u(x,z,t)-U(x,z,t)|\le&\max_{(x,z,t)\in\Omega}\left|\frac{\partial^{N_x+1}u(\varepsilon_x,z,t)}{\partial x^{N_x+1}}\right|\frac{\left|\prod_{i=0}^{N_x}(x-x_i)\right|}{(N_x+1)!}+\max_{(x,z,t)\in\Omega}\left|\frac{\partial^{N_z+1}u(x,\varepsilon_z,t)}{\partial z^{N_z+1}}\right|\frac{\left|\prod_{j=0}^{N_z}(z-z_j)\right|}{(N_z+1)!}+\max_{(x,z,t)\in\Omega}\left|\frac{\partial^{N_t+1}u(x,z,\varepsilon_t)}{\partial t^{N_t+1}}\right|\frac{\left|\prod_{k=0}^{N_t}(t-t_k)\right|}{(N_t+1)!}\\&+\max_{(x,z,t)\in\Omega}\left|\frac{\partial^{N_x+N_z+N_t+3}u(\varepsilon_x^{'},\varepsilon_z^{'},\varepsilon_t^{'})}{\partial x^{N_x+1}\partial z^{N_z+1}\partial t^{N_t+1}}\right|\frac{\left|\prod_{i=0}^{N_x}(x-x_i)\right|\left|\prod_{j=0}^{N_z}(z-z_j)\right|\left|\prod_{k=0}^{N_t}(t-t_k)\right|}{(N_x+1)!(N_z+1)!(N_t+1)!}\end{aligned}\tag{35}$$

where $\Omega=[a,b]\times[c,d]\times[0,T]$. Since the function $u(x,z,t)$ is assumed to be smooth on the interval of approximation, it follows that its derivatives are bounded and thus ∃ constants $C_1,C_2,C_3$ and $C_4$, such that

$$\left.\begin{aligned}&\max_{(x,z,t)\in\Omega}\left|\frac{\partial^{N_x+1}u(x,z,t)}{\partial x^{N_x+1}}\right|\le C_1,\ \max_{(x,z,t)\in\Omega}\left|\frac{\partial^{N_z+1}u(x,z,t)}{\partial z^{N_z+1}}\right|\le C_2,\\&\max_{(x,z,t)\in\Omega}\left|\frac{\partial^{N_t+1}u(x,z,t)}{\partial t^{N_t+1}}\right|\le C_3,\ \max_{(x,z,t)\in\Omega}\left|\frac{\partial^{N_x+N_z+N_t+3}u(x,z,t)}{\partial x^{N_x+1}\partial z^{N_z+1}\partial t^{N_t+1}}\right|\le C_4\end{aligned}\right\}\tag{36}$$

The error bound for Trivariate polynomial interpolation using Chebyshev Gauss-Lobatto nodes on a single domain is governed by the theorem below;

**Theorem 2 (The error bound in a single domain).** The resulting error bound when CGL grid points $\{x_i\}_{i=0}^{N_x}\in[a,b]$ in $x$-variable, $\{z_j\}_{j=0}^{N_z}\in[c,d]$ grid points in $z$-variable, and $\{t_k\}_{k=0}^{N_t}\in[0,T]$ grid points in $t$-variable are used in Trivariate polynomial interpolation is given by

$$E(x,z,t)\le C_1\frac{8\left(\frac{b-a}{4}\right)^{N_x+1}}{(N_x+1)!}+C_2\frac{8\left(\frac{d-c}{4}\right)^{N_z+1}}{(N_z+1)!}+C_3\frac{8\left(\frac{T}{4}\right)^{N_t+1}}{(N_t+1)!}+C_4\frac{8^3\left(\frac{b-a}{4}\right)^{N_x+1}\left(\frac{d-c}{4}\right)^{N_z+1}\left(\frac{T}{4}\right)^{N_t+1}}{(N_x+1)!(N_z+1)!(N_t+1)!}\tag{37}$$

**Proof:** First, using the relation stated in [4] we express Eq. (3.54) as

$$L_{N_x+1}(\hat{x})=(1-\hat{x}^2)T'_{N_x}(\hat{x})=-N_x\hat{x}TN_x(\hat{x})+N_xTN_{x-1}(\hat{x})\tag{38}$$

Using the triangle inequality and noting that $|TN_x(\hat{x})|\le 1,\ \forall\hat{x}\in[-1,1]$, we have

$$|L_{N_x+1}(\hat{x})|=|-N_x\hat{x}T_{N_x}(\hat{x})+N_xTN_{x-1}(\hat{x})|\le|-N_x\hat{x}TN_x(\hat{x})|+|N_xTN_{x-1}(\hat{x})|\le 2N_x\tag{39}$$

The leading coefficient of $L_{N_x+1}(\hat{x})$ is $2^{N_x-1}N_x$, where the components $2^{N_x-1}$ and $N_x$ comes from the leading coefficient of $TN_x(\hat{x})$ and the application of $N_x$-th rule of differentiation on $TN_x(\hat{x})$, respectively. The product factor in the first term of the error bound expression given at Eq. (35) can therefore be taken as the factorized form of monic polynomial $\frac{L_{N_x+1}(\hat{x})}{2^{N_x-1}N_x}$. We write,

$$\prod_{i=0}^{N_x}\left(\hat{x}-\hat{x}_i\right)=\frac{L_{N_x+1}\left(\hat{x}\right)}{2^{N_x-1}N_x},\ \hat{x}\in\left[-1,1\right] \tag{40}$$

Using Eq. (39), it is easy to establish that the monic polynomial Eq. (40) is bounded by

$$\left|\prod_{j=0}^{N_x}\left(\hat{x}-\hat{x}_i\right)\right|=\left|\frac{L_{N_x+1}\left(\hat{x}\right)}{2^{N_x-1}N_x}\right|\le\frac{2N_x}{2^{N_x-1}N_x}=\frac{4}{2^{N_x}} \tag{41}$$

Considering a general interval $x\in\left[a,b\right]$, we can show that the first product factor in Eq. (35) is bounded by

$$\left.\begin{aligned}\max_{a\le\hat{x}\le b}\left|\prod_{j=0}^{N_x}\left(\hat{x}-\hat{x}_i\right)\right|&=\max_{-1\le\hat{x}\le1}\left|\prod_{i=0}^{N_x}\frac{\left(b-a\right)}{2}\left(\hat{x}-\hat{x}_i\right)\right|=\left(\frac{b-a}{2}\right)^{N_x+1}\max_{-1\le\hat{x}\le1}\left|\prod_{j=0}^{N_x}\left(\hat{x}-\hat{x}_i\right)\right|\\&=\left(\frac{b-a}{2}\right)^{N_x+1}\max_{-1\le\hat{x}\le1}\left|\frac{L_{N_x+1}\left(\hat{x}\right)}{2^{N_x-1}N_x}\right|\le\left|\frac{4\left(\frac{b-a}{2}\right)^{N_x+1}}{2^{N_x}}=8\left(\frac{b-a}{4}\right)^{N_x+1}\right.\end{aligned}\right\} \tag{42}$$

Similarly, we conclude that the second and the third product factor are bounded, respectively, by;

$$\max_{c\le z\le d}\left|\prod_{j=0}^{N_x}\left(\hat{z}-\hat{z}_i\right)\right|=\left(\frac{d-c}{2}\right)^{N_z+1}\max_{-1\le\hat{z}\le1}\left|\frac{L_{N_z+1}\left(\hat{z}\right)}{2^{N_z-1}N_z}\right|\le\left|\frac{4\left(\frac{d-c}{2}\right)^{N_z+1}}{2^{N_z}}=8\left(\frac{d-c}{4}\right)^{N_z+1}\right. \tag{43}$$

And

$$\max_{0\le t\le T}\left|\prod_{k=0}^{N_t}\left(t-t_k\right)\right|=\left(\frac{T}{2}\right)^{N_t+1}\max_{-1\le\hat{t}\le1}\left|\frac{L_{N_t+1}\left(\hat{t}\right)}{2^{N_t-1}N_t}\right|\le\left|\frac{4\left(\frac{T}{2}\right)^{N_t+1}}{2^{N_t}}=8\left(\frac{T}{4}\right)^{N_t+1}\right. \tag{44}$$

Using Eqs. (42) −(44), and Eq. (36) in Eq. (35) the proof is completed.

## III. RESULTS AND DISCUSSION

### 3.1 Residual Error Graph

Residual error is useful in accessing the accuracy of numerical approximations where there is no prior knowledge of the exact solutions. We also look at the residual error results in order to ensure that our numerical scheme is accurate. Figure 5.31 show that very accurate results are obtained after the 6th iteration suggesting that the numerical scheme is very accurate and residual errors of order $10^{-8}$ are obtained. Fig2 clearly show the accuracy of the method. The accuracy is seen to improve with an increase in the number of collocation points $N_x$. It is remarkable to note that accurate results with errors of order up to $10^{-8}$ are obtained using very few collocation points in both the $x$ and $t$ variables $N_t\le0$, $N_x\le0$. This is a clear indication that the MDTSCM is powerful method that is appropriate in solving non-linear PDEs. We note also that the TSCM is computationally fast as accurate results are generated in a fraction of a second in the computation considered in this study.

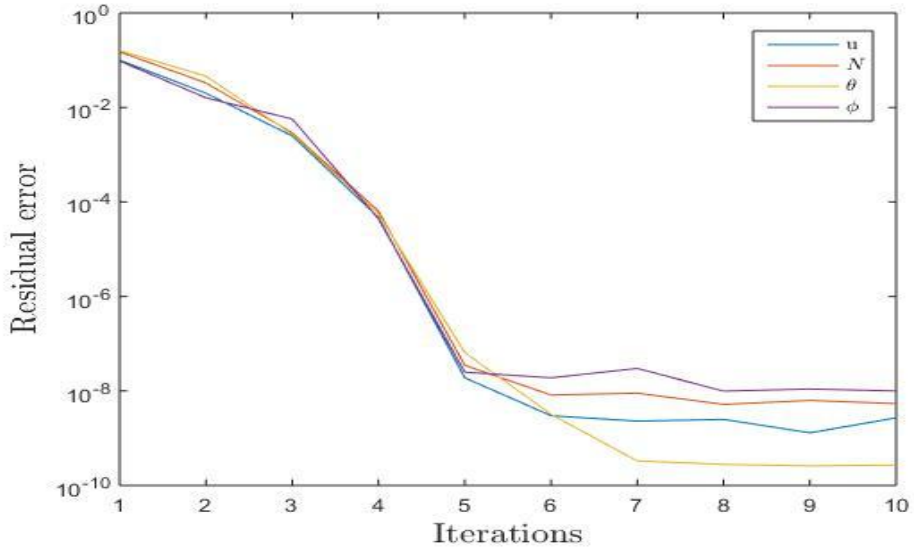


**Figure 2:** Residual error graph for $u, N, \theta,$ and $\phi,$

### 3.2 Convergence Graph

Fig3 depicts that the numerical scheme converges quadratically. Convergence is achieved after only 5 iterations an indicator that the numerical scheme converges faster. The order of convergence is computed and displayed in table 1. Values in this table confirm the order of convergence of the iterative numerical scheme is indeed 2.

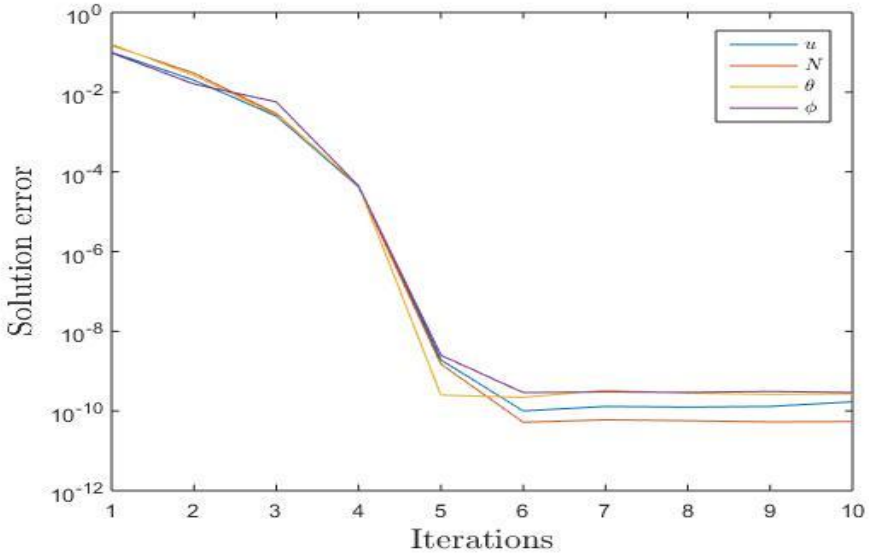


**Figure 3:** Convergence graph for $u, N, \theta,$ and $\phi,$

### 3.3 Order of Convergence

The method will converge if infinity norm errors goes to zero as the number of collocation points increases. It also converges if an increase in collocation points results in a decrease in the infinity norm errors. Table 2 depicts the order of convergence of the MDTSCM for the cases under study. It can be seen that, in the present case considered, the iteration scheme takes about 5 iterations to converge fully. Beyond the point where full convergence is reached, error norm levels off and does not improve with an increase in the number of iterations. This plateau level gives an estimate of the maximum error that can be achieved when using the proposed method with a certain number of collocation points. It is worth remarking that the accuracy of the method depends on the number of collocation points in both the $x$ and $t$ directions.

**Table 1:** Order of Convergence

| **Iterations,** $u$ | **Order of convergence** $\log\left\|\frac{u_{s+1}-u_s}{u_s-u_{s-1}}\right\|$ |
|---|---|
| | |
| 1 | …….. |
| 2 | 1.883 |
| 3 | 1.954 |
| 4 | 1.986 |
| 5 | 1.970 |
| 6 | 1.992 |
| 7 | 2.021 |
| 8 | 1.981 |
| 9 | 1.984 |
| 10 | 1.967 |

So, the order of convergence is quadratic i.e. almost 2 (See above). The rate of convergence of spectral approximations depends only on the smoothness of the solution, yielding the ability to achieve high precision with a small amount of data. The spectral collocation method is chosen considering the fact that it gives an exponential convergence rate, which is very useful in providing highly accurate solutions to nonlinear diff erential equations even if a small number of grid points are used. The spectral methods also work well in solving both linear and nonlinear equations. Generally the spectral methods are computationally less challenging compared to traditional methods but become inaccurate for problems with disjointed coefficients [17].

### 3.4 Stability

The condition number of the matrix measures the ratio of the maximum relative stretching to the maximum relative shrinking that matrix does to any non-zero vectors. The 2-norm condition number of a matrix $A \in \subset^{n\times n}$ is defined as
$\kappa(A) = \|A\|_2 \left\|A^{-1}\right\|_2$.

Without any preconditioning $\kappa(A_n)$ grows proportionally with $n$, which is significantly better than the typical growth of $O(n^{2N})$ in the condition number for the standard tau and collocation methods. It is imperative to note that the condition number of the linear system is independent of the number of collocation points; and the underlying boundary conditions are imposed exactly. Table 2 is a table of condition numbers of coefficient matrices.

**Table 2:** Condition numbers of coefficient matrices

| **Unknown** | **Condition number** |
|---|---|
| $u$ | 3.54778e+03 |
| $N$ | 5.03241e+04 |
| $\theta$ | 2.76865e+03 |
| $\phi$ | 8.05429e+03 |

In view of the sizes of the associated coefficient matrices, above conditions numbers are small, hence the system of linear algebraic system is well posed, and thus the numerical scheme is stable.

To assess the stability of the numerical scheme, we computed condition numbers of the associated coefficient matrices of the system of linear algebraic equations being solved.

## IV. CONCLUSION

In view of size of the small values of condition numbers, we infer that the numerical scheme is stable and thus less sensitive to changes in initial data i.e. the problem is well-posed, and convergence of the iterative numerical scheme is independent of the initial approximation to the solution. The useful features of the iterative numerical method suggest that it is superior and therefore a reliable method for solving this class of fluid dynamics problems.

The multi-domain Trivariate spectral collocation method has been used successfully to solve non-linear parabolic PDEs that arises in wide range of application like genetics, biology, heat and mass transfer and wave processes. The approximate results confirms that the multi-domain Trivariate spectral collocation method is very accurate and is computationally faster. This approach is an alternative to other numerical methods that can be

used to solve non-linear parabolic partial differential equations. The multi-domain Trivariate spectral collocation method being more accurate and computationally faster can therefore be adopted and extended to solve similar problems that model real life phenomenon.

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## REFERENCES

[1] A. H. Khater, R. S. Temsah, Numerical solutions of some nonlinear evolution equations by Chebyshev spectral collocation methods, International Journal of Computer Mathematics, (84), pp. 326-339, 2007.
[2] X. Y. Wang, Exact and explicit solitary wave solutions for the generalized Fishers equation, PhysLett A, (131), pp. 277-291, 1988.
[3] W. Hereman, M. Takaoka , Solitary wave solutions of nonlinear evolution and wave equations using a direct method and MACSYMA, J Phys A , (23), pp. 34-48, 1990.
[4] Martínez, C. A. T., & Fuentes, C. (2017). Applications of Radial Basis Function Schemes to Fractional Partial Differential Equations. In Fractal Analysis-Applications in Physics, Engineering and Technology. IntechOpen.
[5] Lipitakis, A. D. (2019). The Numerical Solution of Singularly Perturbed Nonlinear Partial Differential Equations in Three Space Variables: The Adaptive Explicit Inverse Preconditioning Approach. Modelling and Simulation in Engineering, 2019.
[6] Bustamante Chaverra, C. A., Power, H., Florez Escobar, W. F., & Hill Betancourt, A. F. (2013). Two-Dimensional Meshless Solution of the Non-Linear Convection-Diffusion-Reaction Equation by the Local Hermitian Interpolation Method. Ingeniería y ciencia, 9(17), 21-51.
[7] Barik, N. B., &Sekhar, T. V. S. (2017). A novel RBF-FD meshless scheme in curvilinear geometry for unbounded flows. International Journal for Computational Methods in Engineering Science and Mechanics, 18(4-5), 209-219.
[8] Mai-Duy, N., & Tanner, R. I. (2007). A spectral collocation method based on integrated Chebyshev polynomials for two-dimensional biharmonic boundary-value problems. Journal of Computational and Applied Mathematics, 201(1), 30-47.
[9] Shizgal, B. (2015). Spectral methods in chemistry and physics. Applications to Kinetic Theory and Quantum Mechanics, in: Scientific Computation, Springer, Dordrecht.
[10] Lindvall, K., &Scheffel, J. (2018). A time-spectral method for initial-value problems using a novel spatial subdomain scheme. Cogent Mathematics & Statistics, 5(1), 1-17.
[11] S. S. Motsa, P. Dlamini, M. Khumalo, A new multi-stage spectral relaxation method for solving chaotic initial value systems, Nonlinear Dynamics, Vol. 72, Issue 1-2, pp. 265 283, 2013.
[12] S. S. Motsa, A new piecewise-quasilinearization method for solving chaotic systems of initial value problems, Central European Journal of Physics,10( 4), pp. 936-946, 2012.
[13] S. S. Motsa, A new piece-wise-quasilinearisation method approach to a four-dimensional hyper-chaotic system with cubic nonlinearity, Nonlinear Dynamics. Vol. 70, pp. 651-657, 2012.
[14] Gottlieb, D., & Orszag, S. A. (1977). Numerical analysis of spectral methods: theory and applications, 26. SIAM.
[15] Ben-Yu, G. (1998). Spectral methods and their applications. World Scientific.
[16] Burden, R. L., &Faires, J. D. (2010). Numerical analysis (Ninth edition).
[17] B. Costa, (2004) "Spectral Methods for Partial Diff erential Equations", A Mathematical Journal, 6(4), 1-32.

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