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Stochastic bounds for the Ermakov invariant driven by a Lévy process

E. Cervantes-López [1], E. Méndez-Segura [2]

[1] *General Directorate of Integration, Analysis and Research, National Institute of Statistics and Geography, Av. Héroe de Nacozari Sur #2301, Jardines del Parque, 20276 Aguascalientes, México.*

[2] *Master's Degree in Public Policy, Instituto Tecnológico Autónomo de México, Río Hondo # 1, Colonia Progreso Tizapán, Álvaro Obregón, Zip Code 01080 México City, México.*

ABSTRACT *The work performed by the Russian mathematician V. P. Ermakov on the invariant that has his name was highly referenced in the late seventies when some results were analyzed in different applications of physics related to his research, this generated a wide variety of contributions in classical and quantum physics. The invariant is related to the nonadiabatic Hannayś angle and Berry's phase, and more recently, numerical investigations with additive and multiplicative noise have been carried out on the Ermakov invariant with the purpose of analyzing robustness and behavior. In the present work, stochastic bounds were constructed for the variations that occur when we introduce stochastic noise by a Lévy process on the Ermakov invariant, using a very important theoretical result found by Doney in 2004.*

KEYWORDS *- Invariant, Lévy processes, stochastic limits, physics, Ermakov systems, Lie algebra.*

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1. INTRODUCTION

The main idea is to show through a simulation how we can use the theoretical construction of Doney's theorem for the construction of the bounds in a physical model of the Ermakov invariant with noise of a Lévy process induced in the system that generates the invariant, in this way a pair of stochastic bounds can be constructed supporting the analysis of the behavior of the Ermakov system. In this way, we will use the basic elements that allow us to establish and adequately resolve such an approach.

As a necessity to give an adequate explanation to ordinal differential equations involving continuous stochastic processes, stochastic calculus was developed, a branch of mathematics whose main orientation is on stochastic processes and stochastic differential equations. The application of stochastic calculus in different areas of science, and especially in physics, has been increasingly recurrent, supported by the computational advances that have been developed and that through efficient algorithms have improved the approximations to reality. Theoretical and practical advances in stochastic calculus are increasingly used in various fields such as hydrodynamics, cosmology, spectroscopy, quantum optics, and relativity, as the nature of some physical phenomena requires this type of knowledge. Nowadays, engineering is even using stochastic calculus to generate noise in some processes with discrete or continuous distributions. These types of distributions, discrete and continuous, were related in the 1930s to the generalization of the French mathematician Paul Lévy [1] about the work of Norbert Wiener establishing the development of Lévy processes, named in his honor, which are stochastic processes with stationary and independent increments, and continuous in the probabilistic sense. thus, some Lévy processes are compound Poisson processes, Brownian motion, jumping diffusion processes, and stable processes, for a rigorous query see Barndorff-Nielsen [2]. Lévy processes are important in the areas of finance, economics and recently in applications in physics [3]. Lévy processes have important properties and at first his theory was closely related to the theory of random walks [4] and [5], thus we find that R.A. Doney [6] demonstrated a very interesting theoretical result which states that an arbitrary Lévy process can be bounded by two random walks with identical distributions but different starting points. Through Wiener-Hopf factorization, this result is very important since it can be used in the Ermakov invariant with multiplicative noise and obtain a stochastic bound, which is the goal.

As a first approach to obtaining Ermakov invariants, there are several ways to obtain it Gupta [7] and we work with the method that Korsch [8] and Kaushal and Korsch [9] used to construct the invariants of a variety of time-dependent systems.

The theoretical elements that give mathematical support to the simulations built under the hypothesis that the Ermakov invariant does exist and can be affected by a Lévy process that allows the construction of a stochastic bound that does not depend on the values of the time partition in intervals are established. The document contains the basic elements that involve mathematical, physical, probabilistic, and related stochastic process aspects to support its implementation, for a greater depth in the topics use the established references.

II. THEORY AND METHODS

2.1 Structure constants in the Lie Algebra

The Ermakov invariant can be obtained using several methods, and one of them is the one related to Lie algebras. In 1873, Sophus Lie [10] laid the foundations of Lie's theory by studying the properties of solutions to systems of differential equations and introducing invariants into analysis and differential geometry. Lie showed that the set of continuous transformations, although not globally closed, forms a closed group for composition. This means that, to each group of transformations, we can associate a family of infinitesimal transformations, which contains the information and is associated with Lie algebra, i.e., by considering a family of continuous transformations with $f_i(x_1,\dots,x_n;a_1^0,\dots,a_k^0)$ for each f_i locally defined and depending on k parameters, and a family of continuous transformations with $x_i' = f_i(x_1,\dots,x_n;a_1,\dots,a_k)\quad 1\le i\le n$, taking into account the first order in a Taylor development for each f_i :

$$f_i(x_1,\dots,x_n;a_1^0,\dots,a_k^0+z_k) = x_i + \sum_{j=1}^{k} z_j\, X_{ji}(x_1,\dots,x_n) + \dots \tag{1}$$

results in a transformation that moves points to infinitely small distances :

$$dx_i = \left(\sum_{j=1}^{k} z_j\, X_{ji}(x_1,\dots,x_n)\right) dt \tag{2}$$

In addition, it should be noted that during system integration :

$$\frac{d\zeta_1}{\sum_{j=1}^{k} z_j\, X_{j1}(\zeta_1,\dots,\zeta_n)} = \dots = \frac{d\zeta_n}{\sum_{j=1}^{k} z_j\, X_{jn}(\zeta_1,\dots,\zeta_n)} = dt \tag{3}$$

For each $(z_1,\dots,z_k)$, you get a subgroup of the set of transformations, consisting of a group with a parameter :

$$t \to x_i' = g_i(x_1,\dots,x_n,z_1,\dots,z_k,t) \tag{4}$$

where each

$$x_i = g_i(x_1,\dots,x_n,z_1,\dots,z_k,0) \tag{5}$$

for all i. By reviewing the second-order term in the Taylor development of the functional g_i as a function of t, the following relationships are obtained:

$$\sum_{m=1}^{n}\left(X_{hm}(x)\frac{\partial X_{ji}}{\partial x_m} - X_{jm}(x)\frac{\partial X_{hi}}{\partial x_m}\right) = \sum_{l=1}^{n} c_{lhj}\, X_{li}(x) \tag{6}$$

whereby selecting $z_j = 1$ and $z_h = 0$ when $j \neq h$, Lie gets inphenitesimal transformations that are associated with:

$$A_j f = \sum_{i=1}^{n} X_{ji}(x)\frac{\partial f}{\partial x_i} \tag{7}$$

and rewriting (6) gives a fundamental structure in the vector space generated by the A_j :

$$[A_h, A_j] = A_h A_j - A_j A_h = \sum_{l} c_{lhj}\, A_l(x) \tag{8}$$

where constants c_{lhj} are fixed for each continuous set of transformations and are known as structure constants, associated with Lie algebra, and which are fundamental in the search for the Ermakov invariant.

2.2 Dynamic algebraic approach

Let be a dynamical system described by a configuration space of dimension n, which we denote as M, at a time t. At each point q you can construct a tangent vector space T_qM that is generated by all vector's tangent $\dot{q}$ to all possible curves that pass through it, in addition to being able to construct the velocity phase space TM, known as tangent fibering, and where M and TM are differentiable manifolds. Using the Legendre transform in the dual vector space, known as the phase space, we can construct the cotangent vector space T_q^* whose elements are the moments, and where any function defined in MT_q^* is called a dynamic variable. With the construction of tangent and cotangent space, it is natural to be able to define an arbitrary tensor on the manifold under study and to construct a Lie algebra, which will allow us to find the Ermakov invariant. This algebraic technique [11] considers a Hamiltonian associated with the dynamics of the system. In the case of the forced harmonic oscillator, we find:

$$H(x) = \frac{1}{2m}\left(p^2 + m\omega^2(t)x^2\right) \tag{9}$$

Let be $T_n(p,q)$ a base of functions in phase space that together with a closed form 2 form a Lie algebra, also called dynamic Lie algebra. Writing the Hamiltonian in terms of the functions at the base:

$$H = \sum_n h_n(t) T_n(p,q) \tag{10}$$

where

$$\{T_i, T_m\} = \sum_K C_{im}^k T_k \tag{11}$$

and C_{im}^k is the structure constant and $\{.\}$ is the Poisson bracket. The Ermakov invariant can be expressed as a combination of functions at the base of Lie dynamic algebra:

$$I = \sum_k \lambda_k(t) T_k(p,q) \tag{12}$$

where $\lambda_k(t)$ are the indeterminate coefficients, and from the concept of the dynamics of the invariant:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \{I, H\}_{pq} = 0 \tag{13}$$

generating a system of first-order differential equations:

$$\lambda_k(t) + \sum_n \left[\sum_m C_{nm}^k h_m(t)\right] \lambda_n(t) = 0 \tag{14}$$

with unknown parameters $\lambda_k(t)$. Selecting a basis $T_n(p,q)$ so that, through the Poisson brackets, the structure constants, and the Hamiltonian we can find the unknown parameters λ_k, which allow us to find the Ermakov invariant in a unique way:

$$I = \frac{1}{2}\left[\frac{q^2}{\rho^2} + (q\dot{\rho} - \rho\dot{q})^2\right] \tag{15}$$

2.3 Ermakov systems

The Ermakov system we consider is a pair of coupled oscillators [12]:

$$\ddot{x} + w(t)x = 0 \tag{16}$$

and

$$\ddot{\rho} + w(t)\rho = \frac{\lambda}{\rho^3} \tag{17}$$

where (17) is the Milne-Pinney equation with $\lambda = 1$. The system of equations has been studied in the classical and quantum form; this system of equations is a special type of Newton type with equation of motion:

$$\ddot{x} + \Omega^2(t)x = 0 \tag{18}$$

where $\Omega(t)$ is the time-dependent frequency, which can be written a

$$x(t) = C\rho(t)\sin(\Theta_T(t) + \phi\ C \tag{19}$$

and ϕ are arbitrary constants. The phases are determined by

$$\Theta_T(t) = \int^t \frac{1}{\rho^2(t')} dt' \tag{20}$$

with $\rho(t)$ the solution of (17). In addition, $\Theta_T(t)$ it is decomposed into a dynamic angle and a geometric angle that allow us to analyze the phases of the system:

$$\Delta\Theta_d(t) = \int^t \left[\frac{1}{\rho^2} - \frac{1}{2}\frac{d}{dt'}(\dot{\rho}\rho) + \dot{\rho}^2\right] dt' \tag{21}$$

$$\Delta\Theta_g(t) = \int^t \left[\frac{1}{2}\frac{d}{dt'}(\dot{\rho}\rho) - \dot{\rho}^2\right] dt' \tag{22}$$

These are the necessary elements that have been widely studied in applications in the different branches of theoretical physics, and as shown in the work carried out by Cervantes, Espinoza, Gallegos, and Rosu [13], [14], [15]. The introduction of stochastic noise shows the forced behavior of the dynamics of the system, in the system simulations were carried out on the Ermakov invariant introducing additive and multiplicative stochastic noise, to analyze the robustness of the invariant and the behavior of the dynamic, geometric and total phases, it was found that the main perturbation effects are produced by additive noise. The simulation used a Brownian motion, which is part of the wide variety of Lévy processes.

III. STOCHASTIC BOUNDS FOR A LÉVY PROCESS

Lévy processes began to be studied in 1930, a class of stationary stochastic processes, which is very broad since it includes continuous processes (Brownian motion) and discontinuous processes (with jumps), and where the behavior of the trajectories depends on the density function; this type of stochastic processes only has infinitely divisible density functions associated with them. Important examples of Lévy processes are motion, the Poisson process, the composite Poisson process, stable processes, subordinate processes, to mention a few [16],[17] and [18].

To define a Lévy processes, we assume that $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t>0}, \mathcal{P})$ it is a complete probability space, it is a measure of probability defined over the subsets of which belong to the σ-algebra $\mathcal{F}$; which are the measurable events of the outcome space. A stochastic process X can be seen as a map $(0,\infty) \times \Omega$ to $\mathbb{R}$, and it is established that X_t it is continuous on the right with a limit on the left if there exists a null set N such that if $\omega \notin N$, with $\omega \in \Omega$, then $\lim_{u \downarrow t} X_u(w) = X_t(w)$ for everything t and $\lim_{u \uparrow t} X_u(w)$ exists for all t, then given a process that is continuous on the right with a limit on the left, we have to:

$$\mathrm{X}_{t-} = \lim_{s \uparrow t} X_s \text{ and } \Delta X_t = X_t - \mathrm{X}_{t-} \tag{23}$$

Given a family $(\mathcal{F}_t, t \geq 0)$ of σ-algebras of $\mathcal{F}$ is called a filtration if $\mathcal{F}_s \subseteq \mathcal{F}_t$ when s ≤ t. Let $X = (X(t), t \geq 0)$ be a stochastic process $\mathbb{R}^d$ is defined over a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, then X it is adapted to the filtration $(\mathcal{F}_t, t \geq 0)$ or $\mathcal{F}_t$- adapted if $X(t)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$. In a formal way, the concept can be established.

Definition 1. A process $X = (X(t), t \geq 0)$ is a Lévy process if it is adapted to filtration $(\mathcal{F}_t, t \geq 0)$ and satisfies the following conditions:

(i) $X(0) = 0$ almost certainly

(ii) Each $X(t) - X(s)$ is independent of $\mathcal{F}_s$, $\forall\ 0 \leq s < t < \infty$

(iii) X has stationary increments, i.e., of $X(t) - X(s)$ it has the same distribution of $X(t-s)$, for each $0 \leq s < t < \infty$

(iv) X It is stochastically continuous for everything $\alpha > 0$ and for everything $s \geq 0$
$\lim_{t \to s} P[|X(t) - X(s)| > \alpha] = 0$

Under these conditions, each X Lévy process has a *càdlàg* modification and is itself [1] a Lévy process, so for practical purposes a Lévy process can always be treated as a *càdlàg* process.

The general structure of Lévy processes was gradually developed by De Finetti, Kolmogorov, Lévy and Itô [19], to mention some of the most important. An interesting result is the Lévy-Khintchine formula [20] which gives a characterization of probability measures infinitely divisible through their characteristic functions. Formally, the convolution of two measures of probability, $\mu_i \in \mathrm{M}_1(\mathbb{R}^d)$ where $\mathrm{M}_1(\mathbb{R}^d)$ denotes the set of all Borel measures of probability over $\mathbb{R}^d$, with $i = 1, 2$:

$$(\mu_1 * \mu_2)(A) = \int_{\mathbb{R}^d} 1_A(x+y)\, \mu_1(dx) * \mu_2(dy) \tag{24}$$

where $A \in \mathcal{B}(\mathbb{R}^d)$. It is also defined $(\mu)^{*^n} = \mu * \ldots * \mu$ (n times). Then $\mu \in \mathrm{M}_1(\mathbb{R}^d)$ is infinitely divisible if there is a measure $\mu^{\frac{1}{n}} \in \mathrm{M}_1(\mathbb{R}^d)$ for which $\left(\mu^{\frac{1}{n}}\right)^{*^n} = \mu$, and the characteristic function of $\mu \in \mathrm{M}_1(\mathbb{R}^d)$ is defined as:

$$\phi_\mu(u) = \int_{\mathbb{R}^d} e^{i\langle u,y\rangle}\, \mu(dy) \tag{25}$$

where $\mu \in \mathbb{R}^d$. Now, let ν a measure of Borel be definite on $\mathbb{R}^d \setminus \{0\} = \{x \in \mathbb{R}^d,\ x \neq 0\}$. So ν it's a Lévy measure if:

$$\int_{\mathbb{R}^d \setminus \{0\}} (|y|^2 \wedge 1)\, \nu(dy) < \infty \tag{26}$$

[1] *In mathematics, càdlàg (from the French "continue à droite, limit à gauche" 'continuous to the right, limit to the left'), are functions defined on real numbers, or other types of objects for which there is lateral continuity on the right and for which it is simultaneously assumed that their limits exist on the left at all their points.*

Theorem 1. (Lévy-Khintchine formula). A Borel measure of probability μ over $\mathbb{R}^d$ is infinitely divisible if there exists a vector $b \in \mathbb{R}^d$, a non-negative defined symmetric matrix $A_{(dxd)}$ and a Lévy measure over $\mathbb{R}^d \setminus \{0\}$ such that, for all $\mu \in \mathbb{R}^d$ the characteristic function of μ admits representation:

$$\phi_\mu(u) = e^{\left\{i\langle b,u\rangle - \frac{1}{2}\langle u,Au\rangle + \int_{\mathbb{R}^d\setminus\{0\}}\left[e^{i\langle u,y\rangle} - 1 - i\langle u,y\rangle 1_{\hat{B}(y)}\right]\nu(dy)\right\}} \tag{27}$$

where $\hat{B}(y) = \{y \in \mathbb{R}^d : 0 < |y| < 1\}$.

Conversely, any application of the form (27) is the characteristic function of a measure of probability infinitely divisible over $\mathbb{R}^d$. The triplet (b, A, ν) is called the characteristic of ν.

The relationship between the Lévy-Khintchine formula and Lévy processes is because each of the random variables that make up a Lévy process is infinitely divisible due to the stationarity and independence of the increments. In this way, the distribution of a Lévy process is determined by the form that the Lévy-Khintchine formula can take.

Another result of interest is the decomposition of a Lévy process, which states that each Lévy process $(\mathrm{X(t)}, \mathrm{t} \geq 0)$ has the decomposition of the sample path into continuous and jump parts, this result is known as the Lévy–It$\hat{o}$ theorem.

Theorem 2. If X it is a Lévy process, then there exists $b \in \mathbb{R}^d$, a Brownian motion B_A with a covariance matrix A and an independent Poisson random measure N on $\mathbb{R}^+ x\ (\mathbb{R}^d - \{0\})$ such that, for each $t \geq 0$:

$$X(t) = bt + \mathrm{B}_A(t) + \int_{|x|<1} x\tilde{N}(t,dx) + \int_{|x|\geq 1} xN(t,dx) \tag{28}$$

Thus, we will consider an Ermakov system with (16) and (17) which has an invariant (15), and the dynamics of the geometric and dynamic phases correspond to (21) and (22). Its equivalent stochastic system [21] driven by a Lévy process L_t:

$$dX_t = f(t, X_t)dt + \sigma(t, X_t)dL_t \tag{29}$$

$$X_a = \xi$$

which should be interpreted as the stochastic integral equation:

$$X_t = \xi + \int_a^t f(s, X_s)ds + \int_a^t \sigma(s, X_s)dL(s),\ a \leq t \leq b \tag{30}$$

The case of the parametric oscillator given by equations (16) and (17) was considered, as well as its stochastic matrix formulation:

$$dX_t = \begin{pmatrix} dx \\ d\dot{x} \end{pmatrix}, f(t, X_t) = \begin{pmatrix} \dot{x} \\ -\Omega^2(t)x \end{pmatrix}, \sigma(t, X_t) = \begin{pmatrix} 0 \\ -\alpha_\Omega(t)\rho^m \end{pmatrix} \tag{31}$$

$$d\rho_t = \begin{pmatrix} d\rho \\ d\dot{\rho} \end{pmatrix}, f(t, \rho_t) = \begin{pmatrix} \dot{\rho} \\ -\Omega^2(t)\rho + \dfrac{1}{\rho^3} \end{pmatrix}, \sigma(t, X_t) = \begin{pmatrix} 0 \\ -\alpha_\Omega(t)\rho^m \end{pmatrix} \tag{32}$$

where $\lambda = 1$, α_Ω is the amplitude of the noise frequency Øksendal [22], the parameter m takes the value of zero for additive noise and for the multiplicative case any number greater than unity would be taken, as in Cervantes, Espinoza, Gallegos, and Rosu [13], [14], [15]. By intensifying the amplitude of the noise, we can slightly distort its shape of the Ermakov invariant. Our job is to construct stochastic boundaries to limit the variation that occurs in the invariant by intensifying the noise in the system. Details of the theory and proof of the theorem we will use can be found in the work done by Doney [6], we summarize the most important notation and theory for its computational implementation.

Consider $\prod\{\Re\} > 0$, $X = (X_t, t \geq 0)$ an arbitrary Lévy process and $I = [-\eta_1, \eta_2]$ a fixed interval containing zero, with $\Delta := \prod(I^c) > 0$. Let's do $\tau_0 = 0$ and for $n \geq 1$ we will write τ_n for some time in which J_n, which corresponds to the umpteenth jump in X which it occurs in I^c. Let's consider $\hat{S} = \left(\hat{S}_n, n \geq 0\right)$ the random walk where $\hat{S}_n = X(\tau_n)$ and the random variables $\hat{Y}_1, \hat{Y}_2, \ldots$ for the steps in $\hat{S}$ with $e_r = \tau_r - \tau_{r-1}$ and $r \geq 1$ where $\hat{X}$ is X with the jumps eliminated $J_1, J_2, \ldots$; a Lévy process has been constructed whose measure is constrained from $\prod$ to I, and $\hat{X}$ is independent of $\{(J_n, \tau_n), n \geq 1\}$ and since it has no great jumps, it follows that $E\left(e^{\lambda \hat{X}_t}\right)$ it is finite for all λ real.

Therefore, the contribution of $\sum_1^n \tilde{X}\,(e_r)$ to $\hat{S}_n$ can be easily estimated, and for many purposes $\hat{Y}_r$ it can be replaced by $J_r + \tilde{\mu}$ where $\tilde{\mu} = E\tilde{X}(\tau_1)$. To control the deviation X of $\hat{S}$, it is natural to use stochastic limits. The most natural way to build stochastic bonds is to use the supreme and the infimum, i.e., where each sequence $\widetilde{m}_n$ and $\tilde{\iota}_n$ are independent of $\hat{S}_n$. But this type of approach has some complications depending on each nth iteration, as detailed in the work carried out by Doney [6], in which he showed that there is an alternative way in which the first term $\widetilde{m}_0$ does not depend on each nth term. We should note that for each n fixed, $\left(\hat{S}_n, \widetilde{m}_n\right)$ and $\left(S_n^{(+)}, \widetilde{m}_0\right)$ they have the same distribution, but $\widetilde{m}_0$, as mentioned, does not depend on n and $S_n^{(+)}$are random walks as described in the theorem. Let be:

$I_n := \underset{\tau_n \le t \le \tau_{n+1}}{\mathrm{Inf}} X_t$ (33)

$M_n := \underset{\tau_n \le t \le \tau_{n+1}}{\mathrm{Sup}} X_t$ (34)

where

$I_n = \hat{S}_n + \tilde{\imath}_n, \;\; M_n = \hat{S}_n + \widetilde{m}_n$ (35)

and

$\widetilde{m}_n := \underset{0 \le s < e_{n+1}}{\mathrm{Sup}} \{\tilde{X}(\tau_n + s) - \tilde{X}(\tau_n)\}, \;\; n \ge 1$ (36)

$\tilde{\imath}_n := \underset{0 \le s < e_{n+1}}{\mathrm{Inf}} \{\tilde{X}(\tau_n + s) - \tilde{X}(\tau_n)\}, \qquad n \ge 1$ (37)

Theorem 3. Using the above notation we have, for any $\eta_1, \eta_2 > 0$ fixed with $\Delta := \prod(\mathrm{I^c}) > 0$, $\mathrm{I_n} = \mathrm{S^{(-)}}_{\mathrm{n}} + \tilde{\mathrm{\imath}}_0$, $\mathrm{M_n} = \mathrm{S^{(+)}}_{\mathrm{n}} + \tilde{\mathrm{m}}_0$, $\mathrm{n} \ge 0$ where both the processes $\mathrm{S^+} = \left(\mathrm{S^{(+)}}_{\mathrm{n}}, \mathrm{n} \ge 0\right)$ and $\mathrm{S^{(-)}} = \left(\mathrm{S^{(-)}}_{\mathrm{n}}, \mathrm{n} \ge 0\right)$ are random walks with the same distribution as $\hat{\mathrm{S}}$. In addition, $\mathrm{S^{(+)}}$ and $\tilde{\mathrm{m}}_0$ they are independent, so like also it is $\mathrm{S^{(-)}}$ y $\tilde{\mathrm{\imath}}_0$.

Notice that, for each n fixed, the pairs $\left(\hat{S}_n, \widetilde{m}_n\right)$ and $\left(\mathrm{S^{(+)}} + \tilde{\mathrm{m}}_0\right)$ have the same joint law; however, the latter representation has the great advantage that the term $\tilde{\mathrm{m}}_0$ does not depend on n.

The construction of stochastic limits to a Lévy process was to control these variations and to have a better understanding of the phenomenon, taking it to the practical field helps us to analyze the behavior produced by the effect of noise in a proposed system.

IV. RESULTS.

The presence of noise in nonlinear dynamical systems is of great interest in scientific research since most phenomena evolve in the presence of stochastic noise of different types, motivating the idea of being able to determine the balance between deterministic forces and stochastic factors, in particular, for our case it is important to take into account that models that use Brownian motion have limitations such as the inability to capture jumps, stochastic volatility, and so on. In recent research, there are several types of stochastic noises that we could use to induce in the system, all important, however, we decided to use the generalized hyperbolic distribution that was introduced by Barndorff-Nielsen [23]. The generalized hyperbolic distribution is (HG) has five parameters, i.e., $X \sim HG(\lambda, \alpha, \beta, \delta, \mu)$ where μ is a localization parameter, δ scale parameter, α shape parameter, β asymmetry parameter, λ influences kurtosis and characterizes the classification of HG. The probability density function of the Generalized Hyperbolic Distribution is:

$$\rho_{HG}(x; \lambda, \alpha, \beta, \delta, \mu) = a(\lambda, \alpha, \beta, \delta, \mu)(\delta^2 + (x-\mu)^2)^{\frac{1}{2}\lambda - \frac{1}{4}} \cdot B\left(\lambda - 0.5, \alpha\sqrt{\delta^2 + x^2 - 2x\mu + \mu^2}\right) e^{\beta(x-\mu)} \quad (38)$$

where $a(\lambda, \alpha, \beta, \delta, \mu) = \frac{(\alpha^2 - \beta^2)^{\frac{1}{2}\lambda}}{\sqrt{2\pi}\alpha^{\lambda - \frac{1}{2}}\delta^\lambda B\left(\lambda, \delta\sqrt{\alpha^2 - \beta^2}\right)}$ and $B(\lambda, \cdot)$ y denotes the modified third-type Bessel function with index λ. When selecting $\lambda = \frac{1}{2}$ we have an important subclass due to its wide application in different areas of finance and physics that is known as the Gaussian Inverse Normal Distribution (NIG), the generalized hyperbolic process, which we denote as L_t, is a Lévy process such that $L_1 \sim HG(\lambda, \alpha, \beta, \delta, \mu)$ and has no diffusion component, therefore, it is a pure jump process (they do not have a continuous Brownian component), with infinite variation. The Lévy process is introduced by Barndorff-Nielsen [24] and its distribution function is:

$$\rho_{NIG}(x; \alpha, \beta, \delta, \mu) = \frac{\delta\alpha}{\pi} e^{\delta\sqrt{\delta^2 - \beta^2} - \beta(x-\mu)} \frac{K_1(\alpha g(x-\mu))}{g(x-\mu)} \quad (39)$$

where $x, \mu \in \mathbb{R}, \delta > 0, 0 \le |\beta| \le \alpha$, $g(x) = \sqrt{\delta^2 + x^2}$, and K_1 is a modified Bessel function of the third type see Barndorff-Nielsen and Blaesild [25]. The normal inverse Gaussian Lévy process X_t is defined as a Lévy process with stationary independent increments, where the increments are distributed according to the normal inverse Gaussian distribution and for a Lévy process the characteristic function can be represented as:

$$E(e^{iuX_t}) = e^{t\psi(u)} \quad (40)$$

The Lévy -Khintchine formula for the function in case of the *NIG* Lévy process is given by:

$$\psi(u) = \int_{|x| \ge 1} (1 - e^{iux}) f(x; \alpha, \beta, \delta) dx + \int_{|x| < 1} (1 - e^{iux} - iux) f(x; \alpha, \beta, \delta) dx + iu\gamma \quad (41)$$

where $f(x; \alpha, \beta, \delta)$ is given by $f(x; \alpha, \beta, \delta) = \frac{\alpha\delta}{\pi|x|} e^{(\beta x)} K_1(\alpha|x|)$, $\gamma = \frac{2\alpha\delta}{\pi} \int_0^1 \sinh(\beta x) K_1(\alpha x) dx$ and the normal inverse Gaussian Lévy process is described by the characteristic triplet $(\gamma, 0, \nu)$. Using results from Protter [26] we get a representation of X_t in terms of Poisson processes:

$$X_t = \gamma t + \int_{|y| < 1} y\left(N_t(dy) - t\nu(dy)\right) + \int_{|y| \ge 1} y N_t(dy) \quad (42)$$

where $\nu(dy) = f(y; \alpha, \beta, \delta)dy$. We use the trajectories generated by the probability density function of the normal inverse Gaussian, at various time points. Consider a NIG process, which is a Lévy process L_t and where the increments of L_t are distributed according to the Inverse Gaussian Normal distribution, which contains four parameters $\mu, \delta, \alpha, \beta \in \Re$ with $\delta > 0$ and $0 \leq |\beta| \leq \alpha$. In general, in the different simulations $\delta_t = \delta t$, $\mu_t = \mu t$ and $\gamma = \sqrt{\alpha^2 - \beta^2}$. In the limit case where the variance of the subordinate tends to zero, the NIG process coincides with the Brownian motion and the probability density is normal. For other values of variance, the probability density function of the Normal Inverse Gaussian has an excess of kurtosis and non-zero asymmetry, these excesses allow us to see more clearly the construction of the limits. System (16) and (17) were solved for $\Omega(t) = 2$ with the considerations used in Cervantes, Espinoza, Gallegos, and Rosu [13], [14], [15], also in Schotens [27].
The Ermakov invariant was estimated with a value of 0.49988887 and the stochastics bound were constructed as a variation of the infimum and the supreme.

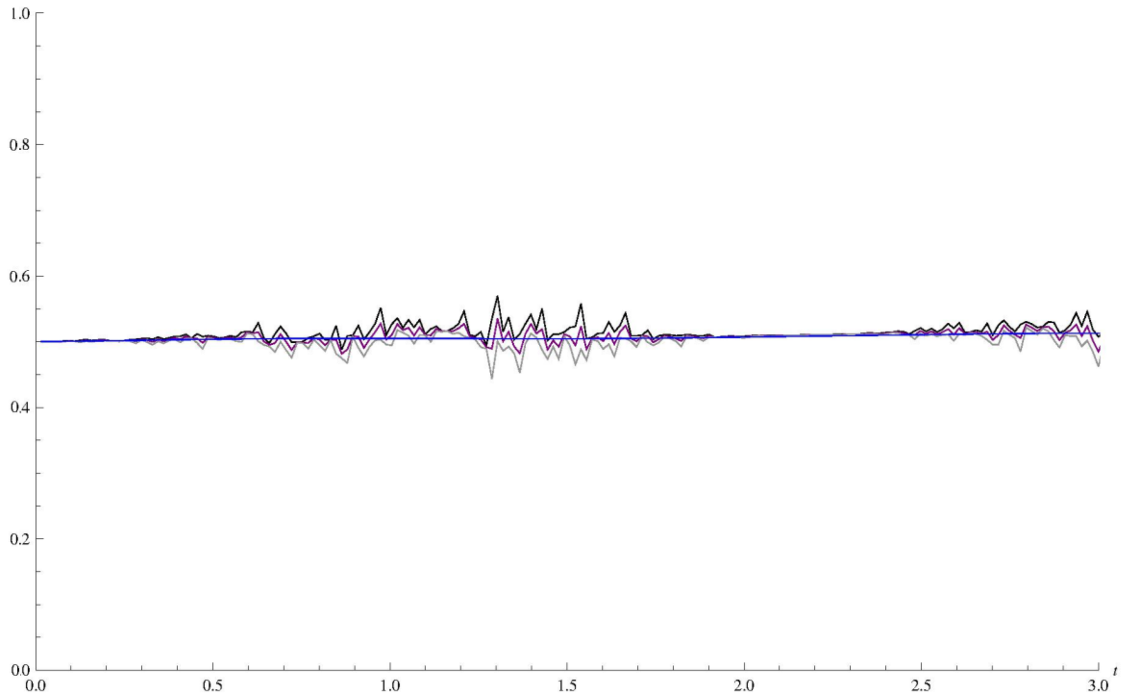

Fig. 1. Dynamic behavior of the stochastic bounds and the Ermakov invariant with $t \in [0, \pi]$, the gray color corresponds to the m = 0 additive noise of amplitude $\alpha_\Omega = 0.15$, lower bound, the magenta color corresponds to the amplitude $\alpha_\Omega = 0.1$ and the black color corresponds to the amplitude $\alpha_\Omega = 0.2$ (uppers bound).

V. CONCLUSIONS

The modeling of systems of stochastic equations is fundamental in pure and applied sciences, especially because of the great advances that have been achieved in computational support, allowing simulations previously not achieved, with this idea, in this research we have managed to build stochastic coefficients with a simpler mathematical structure as are the random walks in function of the infimum and the supremum, in such a way that does not depend on the nth term of the partition of the domain, since as shown by Doney that causes some behavioral problems when n tends to infinity. Now, the Lévy processes are complex and although they allow to generalize the analysis of phenomena with jumps, their mathematical formalization is very rich from the mathematical perspective, at present they continue advancing investigations of the application to different areas of science both theoretical and applied, and undoubtedly that supports enough in the control of models that are simulated through different types of noises, especially the Lévy processes have had important advances and new algorithms have been generated to make more efficient the estimation and not to have problems of convergence as it is shown in several of the established references. It is a first approximation in the application of the theory to a real model that is studied a lot in theoretical physics and that also has its quantum representation, the Ermakov invariant, in this case when analyzing the dynamic behavior affecting the system of equations by a stochastic noise of a Lévy process, different to the Brownian motion, the observation already found that the dynamic invariant is more sensitive to additive noise, as previously observed, is confirmed.

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