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Fuzzy random variables and Kolomogrov's important results

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ABSTRACT :*In this paper an attempt is made to transform Kolomogrov Maximal inequality, Koronecker Lemma, Loeve's Lemma and Kolomogrov's strong law of large numbers for independent, identically distributive fuzzy Random variables. The applications of this results is extensive and could produce intensive insights on Fuzzy Random variables.*

Keywords :*Fuzzy Random Variables, Fuzzy Real Number, Fuzzy distribution function, Strong law of Large Numbers.*

I. Introduction

The theory of fuzzy random variables and fuzzy stochastic processes has received much attention in recent years [1-12]. Prompted for studying law of large numbers for fuzzy random variables is both theoretical since of major concern in fuzzy stochastic theory as in the case of classical probability theory would be the different limit theorems for sequences of fuzzy random variables and practically since they are applicable to statistical analysis when samples or prior information are fuzzy. The concept of fuzzy random variables was introduced by Kwakernack [4] and Puri and Ralesea [6].

In order to make fuzzy random variables applicable to statistical analysis for imprecise data, we need to come up with weak law of large numbers, strong law of large numbers and Kolomogorov inequalities. In the present paper we have deduced Kolomogrov maximal inequality, Kronecker's lemma, and Loeve's Lemma.

2. Preliminaries

In this section, we describe some basic concepts of fuzzy numbers. Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0, 1]$ with the following properties.

1) $\tilde{u}$ is normal, i.e. there exists $x \in R$
2) $\tilde{u}$ is upper semicontinuous.
3) Supp$\tilde{u} = cl\{x \in R : \tilde{u}(x) > 0\}$ is
rights reserved.
4) $\tilde{u}$ is a convex fuzzy set, i.e. $\tilde{u}(\lambda x + (1-\lambda)y) \geq$ min ($\tilde{u}$(x)), $\tilde{u}$ (y)) for $x, y, \in R$ and $\lambda \in [0, 1]$.

Let F(R) be the family of all fuzzy numbers. For a fuzzy set $\tilde{u}$, if we define.

$$L_a\tilde{u} = \begin{cases} \{x: \tilde{u}(x) \geq \propto\}, 0 < \propto \leq 1, \\ \text{Supp } \tilde{u} \propto = 0 \end{cases}$$

Then it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1\tilde{u} \neq \emptyset$ and $L_a\tilde{u}$ is a closed bounded interval for each $\lambda \in [0, 1]$.

From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is a completely determined by the end points of the intervals $L_a\tilde{u} = [u_x^- , u_x^+]$.

3. Fuzzy Random Variables:

Throughout this paper, (Ω, A, P) denotes a complete probability space.

If $\tilde{x} : \Omega \rightarrow F(R)$ is a fuzzy number valued function and *B* is a subset of *R*, then $X^{-1}(B)$ denotes the fuzzy subset of Ω defined by

$$X^{-1}(B)(\omega) = \text{Sup}_{x \in B} \tilde{X}(\omega)(x)$$

for every $\omega \in \Omega$. The function $\tilde{X} : \Omega \rightarrow F(R)$ is called a fuzzy random variable if for every closed subset B or R, the fuzzy set $X^{-1}(B)$ is measurable when consider as a function from Ω to [0,1]. If we denote $\tilde{X}(\omega) = \{ X_x^-(\omega), X_x^{-1}(\omega) \mid 0 \leq \propto \leq 1\}$, then it is a well-known that $\tilde{X}$is a fuzzy random variable if and only if for each $\propto \in$ [0, 1]. X_x^- and $X_x^+$are random variable in the usual sense (for details, see Ref.[11]). Hence, if $\sigma\left(\tilde{X}\right)$is the smallest $\sigma -$ field which makes $\tilde{X}$is a consistent with $\sigma(\{X_x^-, X_x^+ | 0 \leq \propto \leq 1\})$. This enables us to define the concept of independence of fuzzy random variables as in the case of classical random variables.

4. Fuzzy Random Variable and its Distribution Function and Exception

Given a real number, *x*, we can induce a fuzzy number $\tilde{u}$ with membership function $\xi_{\tilde{x}}$ (*r*) such that $\xi_{\tilde{x}}$ (*x*) < 1 for $r \neq x$ (i.e. the membership function has a unique global maximum at x). We call $\tilde{u}$ as a fuzzy real number induced by the real member $\tilde{x}$.

A set of all fuzzy real numbers induced by the real number system the relation ~ on $\mathcal{F}_{\mathbb{R}}$ as $\tilde{x}^{-1} \sim \tilde{x}^{-2}$ if and only if $\tilde{x}^{-1}$and $\tilde{x}^{-2}$ are induce same real number *x*. Then ~ is an equivalence relation, which equivalence classes $[\tilde{x}\ ] = \{\tilde{a} \mid \tilde{a} \sim \tilde{x}\ \}$. The quotient set $\mathcal{F}_{\mathbb{R}}/\sim$ is the equivalence classes. Then the cardinality of $\mathcal{F}_{\mathbb{R}}/\sim$ is equal to the real number system $\mathbb{R}$ since the map $\mathbb{R} \rightarrow \mathcal{F}_{\mathbb{R}}/\sim$ by $x \rightarrow [\tilde{x}\ ]$ is *N*ecall $\mathcal{F}_{\mathbb{R}}/\sim$ as the fuzzy real number system.

Fuzzy real number system $(\mathcal{F}_{\mathbb{R}}/\sim)_{\mathrm{R}}$consists of canonical fuzzy real number we call $\mathcal{F}_{\mathbb{R}}/\sim)_{\mathrm{R}}$ as the canonical fuzzy real number system be a measurable space and $\mathbb{R}, \mathcal{B}$ be a Borel measurable space. $\wp$(**R**) (Power set of **R**) be a set-valued function. According to is called a fuzzy-valued function if {(*x*, *y*) : *y*∈*f*(*x*)} is $\mathcal{M}$ x $\mathcal{B}$. *f*(*x*) is called a fuzzy-valued function if f : X → $\mathcal{F}$ (the set of all numbers). If $\tilde{f}$ is a fuzzy-valued function then $\tilde{f}_x$ is a set-valued function [0, 1]. $\tilde{f}$ is called (fuzzy-valued) measurable if and only if $\tilde{f}_x$ is (set-urable for all ∝∈ [0,1].

Make fuzzy random variables more tractable mathematically, we strong sense of measurability for fuzzy-valued functions. $\tilde{f}$(*x*) be a closed-fuzzy-valued function defined on *X*. From Wu wing two statements are equivalent.

$\tilde{f}_{\propto}^U$ (*x*) are (real-valued) measurable for all $\propto \in$ [0, 1].

fuzzy-valued) measurable and one of $\tilde{f}_{\propto}^L(x)$ and $\tilde{f}_{\propto}^U(x)$ is (real-value) measurable for all ∝∈ [0,1].

A fuzzy random variable called strongly measurable if one of the above two conditions is easy to see that the strong measurability implies measurability. μ) be a measure space and $(\mathbb{R}, \mathcal{B})$ be a Borel measurable space. $\wp(\mathbb{R})$ be a set-valued function. For K ⊆ R the inverse image of *f*.

$$= \{\ x \in X : f(x) \cap K\ \neq \emptyset\ \}$$

u) be a complete σ–finite measure space. From Hiai and Umehaki ing two statements are equivalent.

Borel set K ⊆ $\mathbb{R}$, f^{-1} (*K*) is measurable (i.e. f^{-1} (*K*) ∈ $\mathcal{M}$), $y \in f(x)$ is $\mathcal{M}$ x $\mathcal{B}$ - measurable.

If $\tilde{x}$ is a canonical fuzzy real number then $\tilde{x}_1^{-L} = \tilde{x}_1^U$, Let $\tilde{X}$be a fuzzy random variable. $\tilde{x}_{\propto}^L$ and $\tilde{x}_{\propto}^U$ are random variables for all *x* and $\tilde{x}_1^U$. Let *F*(*x*) be a continuous distribution function of a random variable X. Let $\tilde{x}_{\propto}^{-L}$ and $\tilde{x}_{\propto}^U$have the same distribution function *F*(*x*) for all ∝∈ [0,1]. For any fuzzy observation $\tilde{x}$ of fuzzy random variable $\tilde{X}$ $(\tilde{X}\ (\omega) = \tilde{x}\ )$, the ∝-level set $\tilde{x}_{\ \propto}$ is $\tilde{x}_{\ \propto} = [\tilde{x}_{\propto}^L\ , \tilde{x}_{\propto}^U]$. We can see that $\tilde{x}_{\propto}^L$and$\tilde{x}_{\propto}^U$ are the observations of $\tilde{x}_{\propto}^L$ and $\tilde{x}_{\propto}^U$, respectively. $\tilde{x}_{\propto}^L(\omega) = \tilde{x}_{\propto}^L$ and $\tilde{x}_{\propto}^U$ (ω) = $\tilde{x}_{\propto}^U$ are continuous with respect to ∝ for fixed ω. Thus $\tilde{x}_{\propto}^L$, $\tilde{x}_{\propto}^U$ is continuously shrinking with respect to ∝. Since $[\tilde{x}_{\propto}^L, \tilde{x}_{\propto}^U]$ is the disjoint union of $[\tilde{x}_{\propto}^L, \tilde{x}_{\propto}^L]$ and $(\tilde{x}_1^L, \tilde{x}_{\propto}^U]$ (note that $\tilde{x}_1^L = \tilde{x}_1^U$), for any real number $x \in [\tilde{x}_1^L, \tilde{x}_{\propto}^U]$, we have $x = \tilde{x}_{\beta}^L$ or $F\ (\tilde{x}_{\beta}^U)$ with *x*. If we construct an interval

$A_\propto$ = [min { $\inf_{\propto\leq\beta\leq1}F(\tilde{x}^L_\beta)$, $\inf_{\propto\leq\beta\leq1}F(\tilde{x}^L_\beta)$ }

max {$\sup_{\propto\leq\beta\leq1}F(\tilde{x}^L_\beta)$, $\sup_{\propto\leq\beta\leq1}F(\tilde{x}^L_\beta)$}]

then this interval will contain all of the distributions. (values) associated with each of $x \in [\tilde{x}^L_\propto, \tilde{x}^U_\propto]$, We denote $\tilde{F}(\tilde{x})$the fuzzy distribution function of fuzzy random variable $\tilde{X}$. Then we define the membership function of $\tilde{F}(\tilde{x})$ for any fixed $\tilde{x}$by

$\xi_{\tilde{F}(\tilde{x})}$ (r) = $\sup_{0\leq\propto\leq1}$ ∝ | $_A$, (r)

via the form of "Resolution Identity" . we also say that the fuzzy distribution function $\tilde{F}(\tilde{x})$ is induced by the distribution function $F(x)$. Since $F(x)$ is continuous .we can rewrite A_α as

$A_\propto$ = [min { $\inf_{\propto\leq\beta\leq1}F(\tilde{x}^L_\beta)$, $\min_{\propto\leq\beta\leq1}F(\tilde{x}^L_\beta)$ }

max {$\max_{\propto\leq\beta\leq1}F(\tilde{x}^L_\beta)$, $\max_{\propto\leq\beta\leq1}F(\tilde{x}^L_\beta)$}]

In order to discuss the convergence in distribution for fuzzy random variables in Section 4, we need to claim $\tilde{F}(\tilde{x})$ is a closed-fuzzy-valued function. First of all, we need the following proposition,

We shall discuss the strong and weak convergence in distribution for fuzzy random variables in this section. We propose the following definition.

Definition 3.1 Let $\tilde{X}$ and $\{\tilde{x}_n\}$ be fuzzy random variables defined on the same probability space (Ω $\mathcal{A}$ $\mathcal{P}$).
i) We say that $\{\tilde{X}_n\}$ converges in distribution to $\tilde{X}$level-vise if $(\tilde{x}_n)^L_\alpha$ and $(\tilde{x}_n)^U_\alpha$ converge in distribution to $\tilde{X}^L_\alpha$ and $\tilde{X}^U_\alpha$ respectively for all α. Let $(\tilde{x})$ and $\tilde{F}(\tilde{x})$ be the respective fuzzy distribution functions of $\tilde{X}_\alpha$ and $\tilde{X}$. We say that $\{\tilde{X}_\alpha\}$converges in distribution to $\tilde{X}$ strongly if

$$\lim_{n\to\propto} \tilde{F}_n(\tilde{x}) \overset{s}{\Rightarrow} \tilde{F}(\tilde{x})$$

ii) We say that $\{\tilde{X}_n\}$ converges in distribution to $\tilde{X}$weakly if

$$\lim_{n\to\propto} \tilde{F}_n(\tilde{x}) \overset{w}{\Rightarrow} \tilde{F}(\tilde{x})$$

From the uniqueness of convergence in distribution for usual random variables.We conclude that the above three kinds of convergence have the unique limits.

5. MAIN RESULTS

THEOREM 5.1 (KOLMOGOROU CONVERGENCE THEOREM)

Let {Xn} be independent fuzzy random variables with

EXn = 0 and σ^2_n= EX^2_n<∝, n > 1

If $\sum_{n=1}^{\infty} EX^2_n$<∝ then $\sum_{n=1}^{\infty} X_n$ converges a.s.

Proof : For $\in > 0$

$\in^2$P [|$\sum_{k=n}^{m}$ $\cup_{\propto\in(0,1}$ ∝[($(X_k)\propto$ – v $(X_k)^+_\propto$] ≥ ∈ |]

≤ $\sum_{k=n}^{m}$ E ($\cup_{\propto\in(0,1]}$ ∝[($(X^2_k)^-_\propto$) v $(X^2_k)^+_\propto$]

<$\in^3$ if m, n ≥ n_o

Therefore

$P\ [|\sum_{k=n}^{m}\ \cup_{\propto\in(0,1]}\ \alpha\ [(X_k)_{\propto}^{-} - v\ (X_k)_{\propto}^{+}] \geq \in\ ] \leq \in$

i.e.

$P\ [|\cup_{\propto\in(0,1]}\ \alpha\ [(S_m)_{\propto}^{-} - (S_{n-1})_{\propto}^{-}\ V\ (S_m)_{\propto}^{-}] \geq \in\ ]$

$\leq \in$

if $m,n \geq n_0$

i.e, $\{S_n\}$ is a calculcy sequence in probability.

@ $S_n \xrightarrow{P} S$ Say. Hence by Levys Theorem

$S_n \rightarrow S$ a.s.

i.e. $\sum_{n=1}^{\infty}\ X_n$ conveyes a.s.

Definition 5.1:

Two sequences of Fuzzy random variables $\{X_n\}$

and $\{Y_n\}$ are said to be tail equivalent if

$\sum_{n=1}^{\infty}\ P\ (X_n \neq Y_n) < \alpha$

THEOREM 5.2 :

Suppose that $\{X_n\}_{n=1}^{\alpha}$ be a sequence of independent fuzzy random variables.

Let

$X_n^1 = \begin{cases} X_n & if \quad |X_n| \leq 1 \\ 0 & if \quad |X_n| > 1 \end{cases}$ for all $n \geq 1$

Then the series $\sum X_n$ converges a.s. if the following series converges.

(a) $\sum_{n=1}^{\infty}\ P\ (w : |\ X_n(w)/ > 1\ ) < \infty$

(b) $\sum_{n=1}^{\infty}\ E\ (X_n^1\ ))$ converges and

(c) $\sum_{n=1}^{\infty}\ \sigma_{X_n^1}^2 < \alpha$

Proof

Suppose that the three conditions hold. Because of byKolmogorov Khintchic theorem.

$\sum_{n=1}^{\infty}\ \cup\ \propto[((X_n^1)_{\propto}^{-} - E(X_n^1)_{\propto})\ v\ ((X_n^1)_{\propto}^{+} - E\ (X_n^1)_{\propto}^{+}\ )\ ]$

Conveys a.s. Then (if) implies

$\sum_{n=1}^{\infty}$ $((X_n^1)_\propto^- \vee (X_n^1)_\propto^+$ converges a.s.

By (a) and Borel Can telli lemma

$P(|\cup_{\propto\in(0,1]} \propto [(X_n)_\propto^- \vee (X_n)_\propto^+] > 1 \text{ i.o.}) = 0$

So $|\cup_{\propto\in(0,1]} \propto [(X_n)_\propto^- \vee (X_n)_\propto^+]| \leq 1$ a.s.

Thus

$\sum_{n=1}^{\infty}$ $\cup_{\propto\in(0,1]} \propto [(X_n)_\propto^- \vee (X_n)_\propto^+]$ Converges a.s.

Conversely

if $\sum_n$ $\cup_{\propto\in(0,1]}$ $\propto[(X_n)_\propto^- \vee (X_n)_\propto^+]$

Converges a.s. then the fuzzy random variables

$X_n \rightarrow 0$ a.s.

Hence $P(|\cup_{\propto\in(0,1]}$ $\propto ((X_n)_\propto^- \vee (X_n)_\propto^+) > 1 \text{ i.o.}) = 0$

This implies (a) byBorel Zero one law. Now

$\{X_n\}_{n=1}^{\infty}$and $\{X_n\}_{n=1}^{\infty}$ are tail equivalent sequences

So it is clear that $\sum_{n=1}^{\infty}$ $\cup_{\propto\in(0,1]} \propto [(X_n^1)_\propto^- \vee (X_n^1)_\propto^+]$

converges a.s. when $\sum_{n=1}^{\infty}$ $\cup_{\propto\in(0,1]} \propto [(X_n)_\propto^- \vee (X_n)_\propto^+]$

converges a.s.

Now$[\cup_{\propto\in(0,1]}$ $\propto$ $[(X_n^1)_\propto^- \vee (X_n^1)_\propto^+]_{n=1}^{\infty}$ is a sequence

of uniformly bounded indepdent fuzzy random variables

Let $S_n^1 = \sum_{j=1}^{n}$ $\cup_{\propto\in(0,1]}$ $\alpha\ ((X_j^1)_\propto^- - \vee\ (X_j^1)_\propto^+)$

Since $\sum_{j=1}^{\infty}$ $\cup_{\propto\in(0,1]}$ $\alpha\ [(X_n^1)_\propto^- - (X_n)_\propto^-)]$ converges a.s.

$\lim_{n\to\infty} P(\text{Sup}_{m\geq n}$ $|\cup_{\propto\in(0,1]}$ $\alpha [((S_m^1)_\propto^- - (S_n^1)_\propto^-) \vee ((S_m^1)_\propto^+ - (S_m^1)_\propto^+) \geq \in] = 0$

By the lower bound of Kolmogovovs inequality

$P(\text{Sup}_{m\geq n}$ $|\cup_{\propto\in(0,1]}$ $\alpha [((S_m^1)_\propto^- - (S_n^1)_\propto^-) \vee ((S_m^1)_\propto^+ - (S_m)_\propto^+) \geq \in$

$$\geq 1 - \frac{(2+\in)^2}{\sum_{j=1}^{\infty} \sigma_{X_j^1}^2}$$

Now if

$\sum_{j=1}^{\infty} \sigma_{X_j}^2 = \infty$, then we have

$$P\,(\mathrm{Sup}_{m\geq n}\ \mid \bigcup_{\alpha\in(0,1]} \alpha\,[((S_m^1)_\alpha^- - (S_n^1)_\alpha)^- \ V\ ((S_m^1)_\alpha^+ - (S_m^1)_\alpha^+) \geq \in) = 1$$

This contradicts the contention (1)

So $\sum_{j=1}^{\infty} \sigma_{X_j}^2 < \infty$ proving (c)

The Khintchine – Kolmogoroves theorem implies

$$\sum_n^1 \bigcup_{\alpha\in(0,1]} \alpha\ ((X_n^1)_\alpha^- - E(X_n^1)_\alpha^-))\ V\ ((X_n^1)_\alpha^+ - E(X_n^1)_\alpha^+))$$

converges a.s.

Now Since $\sum_{j=1}^{\infty} X_n^1$ converges a.s.

We have $\sum_{n=1}^{\infty} (X_n^1)$ convergent proving (b)

THEOREM 5. 3 (KOLMOGOROVS INEQUALITY)

Let $X_1\ X_2 \ldots\ldots X_n \ldots.$ be independent fuzzy random variables and

$E(X_i^2)<\infty$, $i \geq 1$. If $S_n = \sum_{i=1}^{n} X_i$ and

$\in > 0$ then

a) $\in^2 P\,(\max_{1\leq k\geq n} \bigcup_{\alpha\in(0,1]} \alpha \mid ((S_k)_\alpha^- - E(S_k)_\alpha^-) \mid V \mid ((S_k)_\alpha^+ - E(S_k)_\alpha^+ \mid) \geq \in \leq \sum_{k=1}^{\infty} \sigma_k^2$

and if moreover

$\bigcup_{\alpha\in(0,1]} \alpha\,[\mid (X_k)^- - (X_k)^+ \mid \leq C < \infty$ a.s.

then

b) $1 - \frac{(2+\in)^2}{\sum_{k=1}^{n} \sigma_k^2} \leq P\,(\max_{1\leq k\geq n} \bigcup_{\alpha\in(0,1]} \alpha\,(\mid (S_k)_\alpha^- - E(S_k)_\alpha^- \mid V \mid ((S_k)_\alpha^+ - E(S_k)_\alpha^+ \mid) \geq \in$

Proof :

We assume $EX_k = 0$, $k \geq 1$.

Define a fuzzy random variables + by

$$+\ \begin{cases} 1st\ k, & n \geq k \geq 1 \\ n+1 & otherwise \end{cases}$$ such that $S_k^2 \geq \in^2$ if there is such a k.

Then $(\max_{k\leq n} \bigcup_{\alpha\in(0,1]} \alpha\,[(S_k)_\alpha^- - V(S_k)_\alpha^+] \geq \in] = [t \leq n]$ and $[t = k] \in -B(x_1 x_2. .x_k)$

Hence

$\int_{[t=k]} \quad \bigcup_{\propto\in(0,1]} \quad \alpha\, [(S_k)^-_\propto ((S_n)^-_\propto - (S_k)^-_\propto) \vee (S_k)^+_\propto ((S_n)^-_\propto - (S_k)^+_\propto))\, d\, P$

$= E \bigcup_{\propto\in(0,1]} \quad \alpha\, [(S_k)^-_\propto I_{t=k} ((S_n)^-_\propto - (S_k)^-_\propto) \vee ((S_n)^+_\propto - (S_k)^+_\propto)]$

$= E\, [\, S_k I_{[t=k]}\; E\; (((S_n)^-_\propto - (S_k)^-_\propto) \vee ((S_n)^+_\propto - (S_n)^+_\propto)]$

$= 0$

Therefore,

$\int_{[t=k]} \quad (S_n)^2_\propto \; d\, P$

$= \int_{[t=k]} \quad ((S_k)^-_\propto + ((S_n)^-_\propto - (S_k)^-_\propto)^2 \;\vee\; ((S_k)^+_\propto + ((S_n)^+_\propto - (S_k)^{+2}_\propto)\, d\, P$

$= \int_{[t=k]} \quad ((S_k)^{-2}_\propto + ((S_n)^-_\propto - (S_k)^-_\propto)^2 \;\vee\; ((S_k)^{+2}_\propto + ((S_n)^+_\propto - (S_n)^{+2}_\propto)$

$+2\, ((S_n)^-_\propto - (S_k)^-_\propto)\; (S_k)^-_\propto) \;\vee\; (S_n)^+_\propto - (S_k)^+_\propto - (S_k)^+_\propto)\, d\, P$

$= \int_{[t=k]} \quad (S_k)^{-2}_\propto - (S_k)^{+2}_\propto)\; d\, P$

$\geq \in^2 P(t = k)$ (5.1)

Therefore

$\in^2 \; P\, (t \leq n) = \in^2 \sum_1^n \quad P\, (t = k)$

$\leq \sum_{k=1}^n \quad \int_{[t=k]} \quad (S_k^-)^2_\propto \;\vee\; (S_n^+)^2_\propto)\; d\, P$

$= \int_{[t\leq n]} \quad (S_k^-)^2_\propto \;\vee\; (S_n^+)^2_\propto)\; d\, P$

$= \int \quad (S_n^-)^2_\propto \;\vee\; (S_n^+)^2_\propto)\; d\, P$

$= E\, (S_n^2)$ (5.2)

But

$ES_n^2 \quad = \quad \sum_{k=1}^n \quad E\, (X_k^-)^2_\propto \;\vee\; (X_k^+)^2_\propto)$

Let X_1 X_2 …… X_n . . . . be independents.

$$\sum_{k=1}^{n} \sigma_k^2$$

So from (3) and (1)

$\in^2 \; P \; [\max_{1\leq k\leq n} \quad S_k \quad \geq \in\,] \leq \sum_{k=1}^n \sigma_k^2$

To prove the lower bound of Kolmogorovs inequality let $f_k = I_{[t>k]}$. Then f_k S_k and X_{k+1} are independent for k = 0, 1, . . . n-1.

Now $[t>k] = [t \leq k]^c \in$ B, $(x_1 \ldots x_k)$ since

$[t \leq k] \in$ B, $(x_1 \ldots x_k)$. Therefore

$E(f_k S_k X_{k+1}) = E(f_k S_k)$

$E(X_{k+1}) = 0$

Now

$$= E((S_k^-)_\propto^2 (f_{k-1})_\propto^- \ V \ (S_k^+)_\propto^2 + (f_{k-1})_\propto^+)$$

$$= E((S_k^-)_\propto^2 (f_{k-1})_\propto^- \ V \ (S_k^+)_\propto^2 + (f_{k-1}^+)_\propto^2)$$

$$= E((S_{k-1}^-)_\propto (f_{k-1}^-)_\propto \ V \ (S_{k-1}^+)_\propto + (f_{k-1}^+)_\propto$$

$$+ (X_k^-)_\propto (f_{k-1}^-)_\propto \ V \ (X_k^+)_\propto + (f_{k-1}^+)_\propto)^2$$

$$= E((S_{k-1}^-)_\propto^2 (f_{k-1}^-)_\propto \ V \ (S_{k-1}^+)_\propto^2 + (f_{k-1}^+)_\propto)$$

$$+ E((X_k^-)_\propto^2 (f_{k-1}^-)_\propto \ V \ (X_k^+)_\propto^2 + (f_{k-1}^+)_\propto)$$

$$= E((S_{k-1}^-)_\propto^2 (f_{k-1}^-)_\propto \ V \ (S_{k-1}^+)_\propto^2 (f_{k-1})_\propto^+)$$

$$+ E((X_k^-)_\propto^2 \ V \ (X_k^+)_\propto^2$$

$$E((f_{k-1}^-)_\propto \ V \ (f_{k-1}^+)_\propto)$$

$$= E((S_{k-1}^-)_\propto^2 (f_{k-1}^-)_\propto \ V \ (S_{k-1}^+)_\propto^2 + (f_{k-1}^+)_\propto)$$

$$+ E((X_k^-)_\propto^2 \ E(f_{k-1}^-)_\propto \ V E((X_k^+)_\propto^2 E(f_{k-1}^+)_\propto)$$

$$= E((S_{k-1}^-)_\propto^2 (f_{k-1}^-)_\propto \ V \ (S_{k-1}^+)_\propto^2 + (f_{k-1}^+)_\propto)$$

$$+ E((X_k^-)_\propto^2 \ V \ (X_k^+)_\propto^2) P(t > k-1) \tag{5.3}$$

Again

$$E((S_k^-)_\propto^2 (f_{k-1}^-)_\propto \ V \ (S_k^+)_\propto^2 + (f_{k-1}^+)_\propto)$$

$$+ E((S_k^-)_\propto^2 (f_k^-)_\propto \ V \ (S_k^+)_\propto^2) (f_k^+)_\propto)$$

$$+ E((S_k^-)_\propto^2 I_{[t=k]} \ V \ (S_k^+)_\propto^2) I_{[t=k]} \tag{5.4}$$

From (5.3) and (5.4)

$$E\,((S_{k-1}^{-})^2_{\propto}(f_{k-1}^{-})_{\propto}\ \ V\ \ (S_{k-1}^{+})^2_{\propto}\ +(f_{k-1}^{+})_{\propto}\ )$$

$$+\ \ E\,((X_k^{-})^2_{\propto}\ \ V\ (X_k^{+})^2_{\propto}\ )\ P\ (t > k\text{-}1)$$

$$=\ E\,((S_k^{-})^2_{\propto}(f_{k-1}^{-})_{\propto}\ V\ (S_k^{+})^2_{\propto}\ )\ (f_{k-1}^{+})_{\propto}\ )$$

$$=\ E\,((S_k^{-})^2_{\propto}(f_k^{-})_{\propto}\quad V\ (S_k^{+})^2_{\propto}\ )\ (f_k^{+})_{\propto}$$

$$+\ \ E\,((S_k^{-})^2_{\propto}\ I_{\ [t=k]}\quad V\ (S_k^{+})^2_{\propto}\ )\ I_{\ [t=k]}$$

Since $|\ (X_k^{-})_{\propto}\ \ V\ (X_k^{+})_{\propto}\ | \leq C$ for all K

and $|\ (X_k^{-})_{\propto} - E\ (X_k^{-})_{\propto}\ \ V\ (X_k^{+})_{\propto} - E\ (X_k^{+})_{\propto}\ | \leq 2C$

$$E\ \ ((S_{k-1}^{-})^2_{\propto}(f_{k-1})_{\propto}\ \ V\ \ (S_{k-1}^{+})^2_{\propto}\ +(f_{k-1})_{\propto}\ )$$

$$+\ \ E\,((X_k^{-})^2_{\propto}\ \ P\ (t \geq k)\ V\ (X_k^{+})^2_{\propto}\ )\ P\ (t \geq k)$$

$$\leq\ E\,((S_k^{-})^2_{\propto}(f_k\ )_{\propto}\ \ V\ \ (S_k^{+})^2_{\propto}(f_k^{-1})_{\propto}\ )$$

$$+\quad (\in +2C)^2\ P\ (t \geq k) \qquad (5.5)$$

Summing over (6) for k=1 to n and after cancellation we get

$$\sum_{k=1}^{n}\quad E\,((X_k^{-})^2\quad V\ (X_k^{+})^2\ )\ P\ (t \geq k)$$

$$\leq\ E\,((S_k^{-})^2_{\propto}(f_n^{-})_{\propto}\ V\ (S_n^{+})^2_{\propto}\ )\ (f_n^{+})_{\propto}\ )$$

$$+\quad (\in +2C)^2\ P\ (t \leq n)$$

Now $S_n^2 f_n^2 \leq\ \in^2$ By definition of t

and $P(t > n) < P(t \geq k)$ if $k \geq n$ imply

$$\sum_{k=1}^{n}\quad E\,((X_k^{-})^2_{\alpha}\ \ P(t \geq k)\qquad V\ E\,((X_k^{+})^2_{\alpha}\ \ P(t \geq n)$$

$$\leq \in^2 E\,((f_n^{-})_{\propto}\ \ V\ (f_n^{+})_{\propto}\ )$$

$$+\ (\in +2C)^2\ P\ (t \leq n)$$

Or

$$\sum_{k=1}^{n}\quad E\,((X_k^{-})^2_{\alpha}\ \ V\ \ ((X_k^{+})^2_{\alpha}\ \ P(t > n)$$

$$\leq \in^2 P(t > n) + (\in +2C)^2\ P\ (t \leq n)$$

$$\leq\ (\in +2C)^2\ P\ (t \leq n)\qquad +\ \ (\in +2C)^2\ P\ (t > n)$$

$= (\in +2C)^2$

Hence

$1 - P\ (t \le n) \le\ (\in +2C)^2 / \sum_{k=1}^{n}\ E\ ((X_k^-)_\alpha^2\ V\ ((X_k^+)_\alpha^2$

$= \frac{(\in+2C)^2}{\sum_{k=1}^{n} \sigma_k^2}$

implies $P\ (t \le n) \ge\ 1 - (\in +2C)^2 / \sum_1^n\ \sigma_k^2$

LEMMA 5.1 : (KRONECKER'S LEMMA)

For sequences $\{a_n\}$ and $\{b_n\}$ of fuzzy real numbers and $\sum_1^\infty$ an converges

and $b_n \uparrow \frac{1}{b_n}$

$\sum_{k=1}^{n}\ b_k\ a_k \to 0$ as $n \to \infty$

Proof : Since $\sum_1^\infty$ an converges $S_n = \sum_1^n\ a_k \rightarrow S$ (say)

$\frac{1}{b_n} \sum_{k=1}^{n}\ [(b_k^-)_\propto (a_k^-)_\propto\ V\ (b_k^+)_\propto (a_k^+)_\propto$

$= \frac{1}{b_n} \sum_{k=1}^{n}\ [(b_k^-)_\propto ((S_k^-)_\propto - (S_{k-1}^-)_\propto$

$-\ (b_k^+)_\propto\ ((S_k^+)_\propto - (S_{k-1}^+)_\propto)]$

$= \frac{1}{b_n}\ (\sum_1^n\ (b_k^-)_\propto (S_k^-)_\propto\ V (b_k^+)_\propto (S_k^+)_\propto$

$-\ \sum_1^n\ (b_k^-)_\propto (S_{k-1}^-)_\propto\ V (b_k^+)_\propto (S_{k-1}^+)_\propto\ )$

$= \frac{1}{b_n}\ (\sum_1^n\ (b_k^-)_\propto (S_k^-)_\propto\ V (b_k^+)_\propto (S_k^+)_\propto$

$-\ \sum_1^{n-1}\ (b_{k+1}^-)_\propto (S_k^-)_\propto\ V (b_{k+1}^+)_\propto (S_k^+)_\propto\ )$

$= \frac{1}{b_n}\ (((b_n^-)_\propto (S_n^-)_\propto\ V (b_n^+)_\propto (S_n^+)_\propto$

$-\ \sum_1^{n-1} ((b_k^-)_\propto - (b_{k+1}^-)_\propto (S_k^-)_\propto\ )$

$V\ ((b_k^+)_\propto - (b_{k+1}^+)_\propto) (S_k^+)_\propto$

$= (S_n^-)_\propto\ V (S_n^+)_\propto \frac{1}{b_n} \sum_1^{n-1}\ ((b_k^-)_\propto - (b_{k+1}^-)_\propto\ ) (S_k^-)_\propto\ )$

$V\ ((b_k^+)_\propto - (b_{k+1}^+)_\propto) (S_k^+)_\propto$

$\rightarrow$ S – S = 0

Since $= \frac{1}{b_n} \sum_1^{n-1} ((b_k^-)_\propto - (b_{k+1}^-)_\propto \, ((S_k^-)_\propto - (S^-)_\propto)$

$V \, ((b_k^+)_\propto - (b_{k+1}^+)_\propto) \, ((S_k^+)_\propto - (S^-)_\propto)$

$\frac{(S^-)_\propto}{(b_n^-)_\propto} \quad \sum_1^{n-1} ((b_k^-)_\propto - (b_{k+1}^-)_\propto \;\; V$

$\frac{(S^+)_\propto}{(b_n^+)_\propto} \quad \sum_1^{n-1} \quad (b_k^+)_\propto - (b_{k+1}^+)_\propto$

$= \frac{1}{b_n} \quad \sum_1^{n-1} [(b_k^-)_\propto - (b_{k+1}^-)_\propto] (S_k^-)_\propto$

$$V \frac{1}{(b_n^+)_\propto} \sum_1^{n-1} [(b_k^+)_\propto - (b_{k+1}^+)_\propto] (S_k^+)_\propto$$

Now

$\frac{(S^-)_\propto}{(b_n^-)_\propto} \quad \sum_1^{n-1} \quad (b_k^-)_\propto - (b_{k+1}^-)_\propto$

$V \frac{(S^+)_\propto}{(b_n^+)_\propto} \sum_1^{n-1} \quad (b_k^+)_\propto - (b_{k+1}^+)_\propto$

$\frac{(S^-)_\propto}{(b_n^-)_\propto} \quad ((b_1^-)_\propto - (b_n^-)_\propto)$

$V \frac{(S^+)_\propto}{(b_n^+)_\propto} \;\; ((b_1^+)_\propto - (b_n^+)_\propto$

$\rightarrow - ((S^-)_\propto \, V \, (S^+)_\propto)$

as $b_n \uparrow \infty$

and

$= \frac{1}{(b_n^-)_\propto} \sum_1^{n-1} ((b_k^-)_\propto - (b_{k+1}^-)_\propto) (\left(S_k^-\right)_\propto - (S_k^-)_\propto$

$V \frac{1}{(b_n^+)_\propto} \sum_1^{n-1} ((b_k^+)_\propto - (b_{k+1}^+)_\propto) ((S_k^+)_\propto - (S^+)_\propto$

$\rightarrow 0$ as $n \rightarrow \infty$

Since

$| \frac{1}{(b_n^-)_\propto} \sum_1^{n-1} ((b_k^-)_\propto - (b_{k+1}^-)_\propto) (\left(S_k^-\right)_\propto - (S_k^-)_\propto$

$V \frac{1}{(b_n^+)_\propto} \sum_1^{n-1} ((b_k^+)_\propto - (b_{k+1}^+)_\propto)((S_k^+)_\propto - (S^+)_\propto)|$

$|\frac{1}{(b_n^-)_\propto} | \sum_{k=1}^{n0} | ((b_k^-)_\propto - (b_{k+1}^-)_\propto) \left(S_k^-\right)_\propto - (S_k^-)_\propto$

$V | \frac{1}{(b_n^+)_\propto} | \sum_{k=1}^{n0} | ((b_k^+)_\propto - (b_{k+1}^+)_\propto)((S_k^+)_\propto - (S^+)_\propto)|$

$|\frac{1}{(b_n^-)_\propto} | \sum_{n0+1}^{n-1} ((b_k^-)_\propto - (b_{k+1}^-)_\propto) (\left(S_k^-\right)_\propto - (S_k^-)_\propto$

$V | \frac{1}{(b_n^+)_\propto} | \sum_{n0+1}^{n-1} ((b_k^+)_\propto - (b_{k+1}^+)_\propto)((S_k^+)_\propto - (S^+)_\propto)|$

for $n > n_0$

$\leq \in \quad + \frac{(b_{n0+1}^-)_\propto (b_n^-)_\propto}{(b_n^-)_\propto}$

$V + \frac{(b_{n0+1}^+)_\propto - (b_n^+)_\propto}{(b_n^+)_\propto} \in$

if $n > n_0$

LEMMA 5.2 : (Loeve)

Let X be a fuzzy random variables and

$q(t) = P\{ | X_\propto^- \ V \ X_\propto^+ | > t \} = 1 - F(t) = \bar{F}(t)$

For every y > 0, x > 0 we have

$x^r \sum_{n=1}^{\infty} q(n^{l/r} x) \leq E(| X_\propto^- |^r \ V \ | X_\propto^- |^r$

$\leq X^r + V X^r \sum_{n=1}^{\infty} q(n^{l/r} x)$

Proof :

$E(| X_\propto^- |^r \ V \ | X_\propto^+ |^r$

$= \int_0^\infty + dP(| X_\propto^- | \ V \ | X_\propto^- |^r \leq t)$

$= -\int_0^\infty + t^r \, dq(t)$

$= -\sum_{n=1}^{\infty} \int_{(n-1)^{1/r} x}^{n^{1/r} x} t^r \, dq(t), \ x > 0$

Now $-\int_{(n-1)^{\frac{1}{r}} x}^{n^{\frac{1}{r}} x} t^r \, dq(t)$

$\leq n x^r [q(n-1)^{\frac{1}{r}} x - q(n^{\frac{1}{r}} x)]$

and

$$-\int_{(n-1)^{\frac{1}{r}}x}^{n^{\frac{1}{r}}x} t^{r}\, dq\,(t)$$

$$\leq (n\text{-}1)x^{r} [\, q\,(n-1)^{\frac{1}{r}}x \,-\, q\,(n^{\frac{1}{r}}x)]$$

If $E\,(|\,X_{\propto}^{-}\,|^{r} \;\text{v}\; |\,X_{\propto}^{+}\,|^{r}\,) = \infty$ the proof is obvious.

and if $E\,(|\,X_{\propto}^{-}\,|^{r} \;\text{v}\; |\,X_{\propto}^{+}\,|^{r}\,) < \infty$ then

$x^{r}\, Nq\;(N^{\frac{1}{r}}x) \to 0$ as $N \to \infty$

In fact,

$$\infty > E\,(|\,X_{\propto}^{-}\,|^{r} \;\text{v}\; |\,X_{\propto}^{+}\,|^{r}\,)$$

$$\geq E\,(|\,X_{\propto}^{-}\,|^{r} \;\text{v}\; |\,X_{\propto}^{+}\,|^{r}\,)\; I\;(|\,X_{\propto}^{-}\,| \;\text{v}\; |\,X_{\propto}^{+}\,| > x\,N^{\frac{1}{r}}]$$

$$= Nx^{r} P\,[\,|\,X_{\propto}^{-}\,|^{r} \;\text{v}\; |\,X_{\propto}^{+}\,|^{r} > N^{\frac{1}{n}}\, x\,]$$

$$= Nx^{r} q\,[\,N^{\frac{1}{r}}\, x\,]$$

If $E\,(|\,X_{\propto}^{-}\,|^{r} \;\text{v}\; |\,X_{\propto}^{+}\,|^{r}\,) < \infty$ by absolute continuity.

of integral $Nx^{r} q\,[\,N^{\frac{1}{r}}\, x\,] \to 0$ as $N \to \infty$

on the other hand.

$$E\,(|\,X_{\propto}^{-}\,|^{r} \;\text{v}\; |\,X_{\propto}^{+}\,|^{r}\,) < \infty$$

$$\geq \sum_{n=1}^{n} (n-1)\, x^{r} [\, q\,(n-1)^{\frac{1}{r}}x - q\,(n^{\frac{1}{r}})x\;]$$

$$-\sum_{n=1}^{N} x^{r}\; [\, q\,(n)^{\frac{1}{r}}x\,] - (N-1)x^{r}\; q\,(n^{\frac{1}{r}})x$$

Since

$Nq\,(n^{\frac{1}{r}}x) \to 0$ the right hand side of the last inequality tends to

$$\sum_{n=1}^{\infty} x^{r}\; q\,(n^{\frac{1}{r}}\, x\,)$$

Now if $\sum_{1}^{\infty} q\,(n^{\frac{1}{r}}\, x\,) < \infty$

then $nq\,(n^{\frac{1}{r}}\, x\,) \to 0$ and

E (| $X_{\propto}^{-}$ |^r v | $X_{\propto}^{+}$ |^r)

$\leq$ $\lim_{N\to\infty}$ $\sum_{n=1}^{N}$ nx^{v} [q $(n-1)^{\frac{1}{r}}x$ – q $(n^{\frac{1}{r}}\,x)$]

$\leq$ $\lim_{N\to\infty}$ x^{v} (1 + $\sum_{1}^{N-1}$ q $(n^{\frac{1}{r}}x)$ – Nq $(n^{\frac{1}{r}}\,x)$)

$\leq$ x^{v} (1 + $\sum_{1}^{\infty}$ q $(n^{\frac{1}{r}}x)$)

which completes the proof.

THEOREM 5.4: (KOLMOGOROVS STRONG LAW OF LARGE NUMBERS for independent identicals distributed r.v.s.)

Let {x_n} be a sequence of indendent identically distributed fuzzy random variables then

$\frac{S_n}{n} \to C < \infty$ a.s.

if and only if E (| $(X_1)_{\propto}^{-}$ v | $(X_1)_{\propto}^{-}$) <∞

and then C = E (X_1)

Proof

For the only if part let A_n = (| $X_{\propto}^{-}$ | v | $X_{\propto}^{+}$) ≥ n

then $\sum_{1}^{\infty}$ ∑ E (| $(X_1)_{\propto}^{-}$ v | $(X_1)_{\propto}^{-}$ + P $\sum_{1}^{\infty}$ (An) (5.7)

Now P(An) = P (| $X_{\propto}^{-}$ | v | $X_{\propto}^{+}$ ≥ n)

= P (| $(X_1)_{\propto}^{-}$ v | $(X_1)_{\propto}^{+}$ ≥ n)

$\frac{S_n}{n} \xrightarrow{a.s.} < \infty$

then $\frac{(X_n)_{\propto}^{-}}{n}$ V $\frac{(X_n)_{\propto}^{+}}{n}$ = $\frac{(S_n)_{\propto}^{-}}{n}$ V $\frac{(S_n)_{\propto}^{+}}{n}$

– $\frac{(n-1)}{n}\,\frac{(S_{n-1})_{\propto}^{-}}{n-1}\,\frac{(S_{n-1})_{\propto}^{+}}{n-1}$

$\to$ $C - C = 0$ $a.s.$

Hence P (|$\frac{(X_1)_{\propto}^{-}}{n}$ | V | $\frac{(X_1)_{\propto}^{+}}{n}$ >$\frac{1}{2}$ i.o.)

By Borel 0 – 1 Law $\sum_{n=1}^{\infty}$ P (| $(X_n)_{\propto}^{-}$ | v | $(X_n^{+})_{\propto}$ ≥ $\frac{n}{2}$)

i.e. $\sum_{n=1}^{\infty}$ P (| $(X_1)_{\propto}^{-}$ v $(X_1)_{\propto}^{+}$) ≥ $\frac{n}{2}$) <∞

$\Rightarrow \infty > \sum_{n}$ P [$|(X_1)_{\propto}^{-}$ v $(X_1)_{\propto}^{+}$) $\geq \frac{n}{2}$)

$\geq \sum_{n}$ P (A_n)

$\Rightarrow \sum_{n}$ P (A_n) $<\infty$

So from (1) E (| $(X_1)_{\propto}^{-}$ v $(X_1)_{\propto}^{+}$)) $<\infty$

Conversely let

E (($X_1)_{\propto}^{-}$ v $(X_1)_{\propto}^{+}$) $<\infty$

and C = E (($X_1)_{\propto}^{-}$ v $(X_1)_{\propto}^{+}$)

Define $(X_k^{-})_{\propto}^{*}$ V $(X_k^{+})_{\propto}^{*}$

= $((X)_{\propto}^{-}$ v $(X)_{\propto}^{+})$ I [| X_k | $\leq$ k] k=1,2,3, . . .

and $(S_n)_{\propto}^{-*}$ V $(S_n)_{\propto}^{+*}$

= $((X)_{\propto}^{-*}$ v $(X)_{\propto}^{+*}$ $+ \left(X_2^{-}\right)_{\propto}^{*}$ v $\left(X_2^{-}\right)_{\propto}^{+*}$

$+ \ldots . \left(X_n^{-}\right)_{\propto}^{*}$ v $(X_2^{+})_{\propto}^{*}$

Then X_k, k=1, 2, ... n are independent

and $|(X_k)_{\propto}^{*}$ v $(X_k)_{\propto}^{*}$ $| \leq$ k

Now

$\sum_{k=1}^{\infty}$ P [$(X_k)_{\propto}^{-}$ v $(X_k)_{\propto}^{+}$ $\neq (X_k)_{\propto}^{-*}$ v $(X_k)_{\propto}^{+*}$

= $\sum_{1}^{\infty}$ P ($|(X_k)_{\propto}^{-}$ v $(X_k)_{\propto}^{+}$ | > k)

$\leq \sum_{1}^{\infty}$ P (A_k) $<\infty$

@ P [$(X_k)_{\propto}^{-}$ v $(X_k)_{\propto}^{+}$ $\neq (X_k^{-})_{\propto}^{-*}$ v $(X_k)_{\propto}^{+*}$ i.o.

Hence $\frac{S_n}{n}$ and $\frac{S_n^{*}}{n}$ trends to the same limit a.s. if they converge at all in.

$$\frac{(S_n)_{\propto}^{-} \text{ v } (S_n)_{\propto}^{+} - (S_n^{*})_{\propto}^{-} \text{ v } (S_n^{*})_{\propto}^{+}}{n} \rightarrow 0 \ a.s.$$

$$as \ \ n \rightarrow \infty$$

So it is enough to prove that $\frac{S_n^*}{n} \to E\left((X_1)_\propto^- \vee (X_1)_\propto^+\right) < \infty$

Now X_n^* are independent but may be necessarily be identically distributed, we shall show that

$$\sum_{n=1}^{\infty} \frac{\sigma^2 \left((X_n)_\propto^{-*} \vee (X_n)_\propto^{+*}\right)}{n} < \infty$$

and that will imply $\frac{S_n^*}{n} \to E\left(\frac{S_n^*}{n}\right)$ converges to zero almost surely.

$E\left((X_n)_\propto^{-*} \vee (X_n^+)_\propto^*\right)$

$= E\left((X_n)_\propto^+ \vee (X_n)_\propto^-\right) \; I\left[\,|(X_n)_\propto^- \vee (X_n)_\propto^+) \le n\,\right]$

$= E\left((X_1)_\propto^- \vee (X_n)_\propto^+\right) \; I\left[\,|(X_1)_\propto^- \vee (X_n)_\propto^+) \le n\,\right]$

$\to E\left((X_1)_\propto^- \vee (X_1)_\propto^+\right)$ G.S.

Therefore

$E\left(\frac{(S_n)_\propto^{-*} \vee (S_n)_\propto^{+*}}{n}\right) \to E\left(|(X_1)_\propto^- \vee (X_1)_\propto^+\right)$

$\sum_1^{\infty} \sigma^2 \left(\frac{(X_n)_\propto^{-*}}{n} \vee \frac{(X_n)_\propto^{+*}}{n}\right)$

$\le \sum_1^{\infty} E\left(\frac{(X_n^-)_\propto^{*2} \vee (X_n^+)_\propto^{*2}}{n^2}\right)$

$= \sum_1^{\infty} \frac{1}{n^2} \int_{[|(X_1)_\propto^- \vee (X_1)_\propto^+ \le n]} \left((X_n)_\propto^{-2} \vee (X_n)_\propto^{+2}\right) dP$

$= \sum_1^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \int_{[k-1 < |(X_n)_\propto^- \vee (X_n)_\propto^+] \le k} \left((X_n)_\propto^{-2} \vee (X_n)_\propto^{+2}\right) dP$

$= \sum_1^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \int_{[k-1 < |(X_1)_\propto^- \vee (X_1)_\propto^+] \le k} \left((X_1)_\propto^{-2} \vee (X_1)_\propto^{+2}\right) dP$

$= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{1}{n^2} \int_{[k-1 < |(X_1)_\propto^- \vee (X_1)_\propto^+] \le k} \left((X_1)_\propto^{-2} \vee (X_1)_\propto^{+2}\right) dP$

$\le 2 \sum_{k=1}^{\infty} \frac{1}{k} k^2 P\left[k-1 < |(X_1)_\propto^- \vee (X_1)_\propto^+\right] \le k\,]$

$= 2 \sum_{k=1}^{\infty} \quad \text{k P} [(\text{k} - 1) < |(X_1)_{\propto}^{-} \ \text{v} \ \left(X_1\right)_{\propto}^{+} < k]$

$= 2 \sum_{k=1}^{\infty} \quad (\text{k}–1)\text{P} [\text{k} - 1 < (|(X_1)_{\propto}^{-} \ \text{v} \ \left(X_1\right)_{\propto}^{+} \leq k] + 2$

$\leq 2 \sum_{1}^{\infty} \quad \int_{[\text{k}-1 < (|(X_1)_{\propto}^{-} \ \text{v} \ \left(X_1\right)_{\propto}^{+}] \leq k} \quad ((X_1)_{\propto}^{-} \ V \ (X_1)_{\propto}^{+}) \ +2$

$= 2 (\text{E} (|(X_1)_{\propto}^{-} \ \text{v} \ \left(X_1\right)_{\propto} + 1$

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