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Certain D - Operator for Srivastava H_B - Hypergeometric Functions of Three Variables

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Abstract : *The aim of this paper is to derive certain relations involving Srivastava's hypergeometric functions H_B in three variables. Many operator identities involving these pairs of symbolic operators are first constructed for this purpose. By means of these operator identities, which express the aforementioned H_B - hypergeometric functions in terms of such simpler functions as the products of the Gauss and Appell hypergeometric functions. Other closely-related results are also considered briefly. Also, we have derived certain new integral representations for the H_B - hypergeometric functions of three variables defined earlier by Lauricella [8] and Srivastava [14] .*

Keywords: *Decomposition formulas; Srivastava's hypergeometric functions; Multiple hypergeometric functions; Gauss hypergeometric function; Appell's hypergeometric functions; Generalized hypergeometric function.*
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I. Introduction

A great interest in the theory of multiple hypergeometric functions (that is, hypergeometric functions of several variables) is motivated essentially by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric functions (see, for details, [17]; see also the recent works [11,12] *and* the references cited therein). For instance, the energy absorbed by some nonferromagnetic conductor sphere included in an internal magnetic field can be calculated with the help of such functions [12].

Hypergeometric functions of several variables are used in physical and quantum chemical applications as well (cf. [14,16]). Especially, many problems in gas dynamics lead to solutions of degenerate second-order partial differential equations which are then solvable in terms of multiple hypergeometric functions. Among examples, we can cite the problem of adiabatic flat-parallel gas flow without whirlwind, the flow problem of supersonic current from vessel with flat walls, and a number of other problems connected with gas flow [5].

We note that Riemann's functions and the fundamental solutions of the degenerate second order partial differential equations are expressible by means of hypergeometric functions of several variables [6]. In investigation of the boundary-value problems for these partial differential equations.

The familiar operator method of Burchnall and Chaundy (cf. [2,3]; see also [4]) has been used by them rather extensively for finding recurrence formulas for hypergeometric functions of two variables in terms of the classical Gauss hypergeometric function of one variable.

Lauricella [8] actually defined the ten triple hypergeometric functions F_E , F_F , ...F_R in addition, of course, to his four functions F_A , F_B , F_C and F_D of three (or n) variables, Srivastava [14] added three new functions F_A , F_B and F_C to the Lauricella set of function hypergeometric functions of three variables, here we shall obtain the integral representations of H_B - hypergeometric functions in quite a different form.

S*uppose that* a hypergeometric function in the form (c.f. [4,13])

(1.1) $$ {}_2F_1(\alpha , \beta ; \gamma ; z) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!(\gamma)_n} z^n $$

for γ neither zero nor a negative integer.

Now we consider H_B - hypergeometric function defined in ([17] as follows :

(1.2) $$ H_B = H_B\,(\alpha_1, \alpha_2, \alpha_3; \gamma_1, \gamma_2, \gamma_3; z_1, z_2, z_3) $$

$$= \sum_{n_1,n_2,n_3=0}^{\infty} \frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

which were introduced and investigated, over four decades ago, by Srivastava (see, for details, [14,15]; see also [17, p. 43] and [18, pp. 68–69]). Here, and in what follows

$$(\lambda)_\mu = \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)}$$

denotes the Pochhammer symbol (or the shifted factorial) for all admissible (real or complex) values of λ and μ. *Also*, we study the H_B - hypergeometric function, where it is regular in the unit hypersphere (c.f. [2,3]), for the H_B - function, we can define as contiguous to it each of the following functions, which are samples by uppering or lowering one of the parameters by unity.

(1.3) $$H_B(\alpha+) = \sum_{n_1,n_2,n_3=0}^{\infty} \frac{\alpha_1+n_1+n_3}{\alpha_1} \frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

(1.4) $$H_B(\alpha-) = \sum_{n_1,n_2,n_3=0}^{\infty} \frac{\alpha_1}{\alpha_1-1+n_1+n_3} \frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

(1.5) $H_B(\alpha+,\beta+)$

$$= \sum_{n_1,n_2,n_3=0}^{\infty} \frac{\alpha_1+n_1+n_3}{\alpha_1} \frac{\alpha_2+n_1+n_2}{\alpha_2} \frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3},$$

(1.6) $$D = \sum_{j=1}^{3} d_j \;,\; d_j = z_j \frac{\partial}{\partial z_j}$$

and the way we effect it with the recursions relations as it is found in the second part of the research..
We obtain, as a result of acting by D on this function a differential equation, some special cases for a group of differential equations are the functions that are effected by the differential operator. There is a numerical example for one of these cases.

II. A Set Of Operator Of H_B - Hypergeometric Function

By applying the operator D in (1.6) to (1.2), we find the following set of operator identities involving the Gauss function ${}_2F_1$, the Appell functions, and Srivastava's hypergeometric functions H_B defined by (1.2) is

(2.1) $$\mathrm{D}H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;z_1,z_2,z_3)$$

$$= \mathrm{D}\sum_{n_1,n_2,n_3=0}^{\infty} \frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

$$= \sum_{n_1,n_2,n_3=0}^{\infty} \frac{(n_1+n_2+n_3)(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

$$= \sum_{n_1,n_2,n_3=0}^{\infty} \frac{(\alpha_1)_{n_1+1+n_3}(\alpha_2)_{n_1+1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1+1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1+1}(z_2)^{n_2}(z_3)^{n_3}$$

$$+ \sum_{n_1,n_2,n_3=0}^{\infty} \frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2+1}(\alpha_3)_{n_2+1+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2+1}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2+1}(z_3)^{n_3}$$

$$+ \sum_{n_1,n_2,n_3=0}^{\infty} \frac{(\alpha_1)_{n_1+n_3+1}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3+1}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3+1}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3+1}$$

$$= (z_1)\sum_{n_1,n_2,n_3=0}^{\infty} \frac{(\alpha_1+n_1+n_3)(\alpha_2+n_1+n_2)}{(\gamma_1+n_1)} \frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

$$+(z_2)\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_2+n_1+n_2)(\alpha_3+n_2+n_3)}{(\gamma_2+n_2)}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

$$+(z_3)\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_1+n_1+n_3)(\alpha_3+n_2+n_3)}{(\gamma_3+n_3)}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

$$=(z_1)\frac{\alpha_1\alpha_2}{\gamma_1}\sum_{n_1,n_2,n_3=0}^{\infty}\frac{\gamma_1(\alpha_1+n_1+n_3)(\alpha_2+n_1+n_2)}{\alpha_1\alpha_2(\gamma_1+n_1)}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

$$+(z_2)\frac{\alpha_2\alpha_3}{\gamma_2}\sum_{n_1,n_2,n_3=0}^{\infty}\frac{\gamma_2(\alpha_2+n_1+n_2)(\alpha_3+n_2+n_3)}{\alpha_2\alpha_3(\gamma_2+n_2)}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

$$+(z_3)\frac{\alpha_1\alpha_3}{\gamma_3}\sum_{n_1,n_2,n_3=0}^{\infty}\frac{\gamma_3(\alpha_1+n_1+n_3)(\alpha_3+n_2+n_3)}{\alpha_1\alpha_3(\gamma_3+n_3)}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

i.e.

(2.2)
$$DH_B=\frac{\alpha_1\alpha_2z_1}{\gamma_1}H_B(\alpha_1+1,\alpha_2+1,\alpha_3;\gamma_1+1,\gamma_2,\gamma_3;z_1,z_2,z_3)$$
$$+\frac{\alpha_2\alpha_3z_2}{\gamma_2}H_B(\alpha_1,\alpha_2+1,\alpha_3+1;\gamma_1,\gamma_2+1,\gamma_3;z_1,z_2,z_3)$$
$$+\frac{\alpha_1\alpha_3z_3}{\gamma_3}H_B(\alpha_1+1,\alpha_2,\alpha_3+1;\gamma_1,\gamma_2,\gamma_3+1;z_1,z_2,z_3)\ .$$

From which and using the contiguous functions relations (1.3), (1.4), (1.5), (1.6) we have

(2.3)
$$DH_B=\frac{\alpha_1\alpha_2z_1}{\gamma_1}H_B(\alpha_1+,\alpha_2+;\gamma_1+)+\frac{\alpha_2\alpha_3z_1}{\gamma_2}H_B(\alpha_2+,\alpha_3+;\gamma_2+)$$
$$+\frac{\alpha_1\alpha_3z_1}{\gamma_3}H_B(\alpha_1+,\alpha_3+;\gamma_3+)$$

i.e. the partial differential equation

$$\left[DH_B-\frac{\alpha_1\alpha_2z_1}{\gamma_1}H_B(\alpha_1+,\alpha_2+;\gamma_1+)-\frac{\alpha_2\alpha_3z_1}{\gamma_2}H_B(\alpha_2+,\alpha_3+;\gamma_2+)-\frac{\alpha_1\alpha_3z_1}{\gamma_3}H_B(\alpha_1+,\alpha_3+;\gamma_3+)\right]=0$$

has a solution in the form : -

$$H_B\left(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;z_1,z_2,z_3\right)=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}\ .$$

Also, we can write the following recursion relations H_B - hypergeometric function as follows:

(2.4)
$$d_1H_B\left(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;z_1,z_2,z_3\right)$$
$$=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(n_1)(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$
$$=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_1)_{n_1+1+n_3}(\alpha_2)_{n_1+1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1+1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1+1}(z_2)^{n_2}(z_3)^{n_3}$$
$$=(z_1)\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_1+n_1+n_3)(\alpha_2+n_1+n_2)}{(\gamma_1+n_1)}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$
$$=\frac{\alpha_1\alpha_2z_1}{\gamma_1}H_B(\alpha_1+1,\alpha_2+1,\alpha_3;\gamma_1+1,\gamma_2,\gamma_3;z_1,z_2,z_3)$$

(2.5) $$d_2 H_B\left(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;z_1,z_2,z_3\right)$$

$$=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(n_2)(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

$$=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2+1}(\alpha_3)_{n_2+1+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2+1}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2+1}(z_3)^{n_3}$$

$$=(z_2)\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_2+n_1+n_2)(\alpha_3+n_2+n_3)}{(\gamma_2+n_2)}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

$$=\frac{\alpha_2\alpha_3 z_2}{\gamma_2}H_B(\alpha_1,\alpha_2+1,\alpha_3+1;\gamma_1,\gamma_2+1,\gamma_3;z_1,z_2,z_3)$$

(2.6) $$d_3 H_B\left(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;z_1,z_2,z_3\right)$$

$$=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(n_3)(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

$$=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_1)_{n_1+n_3+1}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3+1}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3+1}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3+1}$$

$$=(z_3)\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_1+n_1+n_3)(\alpha_3+n_2+n_3)}{(\gamma_3+n_3)}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

$$=\frac{\alpha_1\alpha_3 z_3}{\gamma_3}H_B(\alpha_1+1,\alpha_2,\alpha_3+1;\gamma_1,\gamma_2,\gamma_3+1;z_1,z_2,z_3)\ .$$

III. Some Recurrence Relations For H_B - Hypergeometric Function

3.1 Putting $\alpha_1=\alpha_2$ and $\gamma_1=\gamma_2$ in (2.4), (2.5), and (2.6) we have

(3.1) $$d_1 H_B\left(\alpha_1,\alpha_1,\alpha_3;\gamma_1,\gamma_1,\gamma_3;z_1,z_2,z_3\right)$$

$$=\frac{\alpha_1^2 z_1}{\gamma_1}H_B(\alpha_1+1,\alpha_1+1,\alpha_3;\gamma_1+1,\gamma_1,\gamma_3;z_1,z_2,z_3)$$

(3.2) $$d_2 H_B\left(\alpha_1,\alpha_1,\alpha_3;\gamma_1,\gamma_1,\gamma_3;z_1,z_2,z_3\right)$$

$$=\frac{\alpha_1\alpha_3 z_2}{\gamma_1}H_B(\alpha_1,\alpha_1+1,\alpha_3+1;\gamma_1,\gamma_1+1,\gamma_3;z_1,z_2,z_3)$$

(3.3) $$d_3 H_B\left(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;z_1,z_2,z_3\right)$$

$$=\frac{\alpha_1\alpha_3 z_3}{\gamma_3}H_B(\alpha_1+1,\alpha_1,\alpha_3+1;\gamma_1,\gamma_1,\gamma_3+1;z_1,z_2,z_3)\ .$$

3.2 Putting $\alpha_1=\alpha_3$ and $\gamma_1=\gamma_3$ in (2.4), (2.5), and (2.6) we have

(3.4) $$d_1 H_B\left(\alpha_1,\alpha_2,\alpha_1;\gamma_1,\gamma_2,\gamma_1;z_1,z_2,z_3\right)$$

$$=\frac{\alpha_1\alpha_2 z_1}{\gamma_1}H_B(\alpha_1+1,\alpha_2+1,\alpha_1;\gamma_1+1,\gamma_2,\gamma_1;z_1,z_2,z_3)$$

(3.5) $$d_2 H_B\left(\alpha_1,\alpha_2,\alpha_1;\gamma_1,\gamma_2,\gamma_1;z_1,z_2,z_3\right)$$

$$=\frac{\alpha_2\alpha_1 z_2}{\gamma_2}H_B(\alpha_1,\alpha_2+1,\alpha_1+1;\gamma_1,\gamma_2+1,\gamma_1;z_1,z_2,z_3)$$

(3.6) $$d_3 H_B\left(\alpha_1,\alpha_2,\alpha_1;\gamma_1,\gamma_2,\gamma_1;z_1,z_2,z_3\right)$$
$$=\frac{\alpha_1^{\,2} z_3}{\gamma_1} H_B(\alpha_1+1,\alpha_2,\alpha_1+1;\gamma_1,\gamma_2,\gamma_1+1;z_1,z_2,z_3).$$

3.3 Some special cases for the H_B - hypergeometric function which given us some differential equations its functions as follows:

1- The function

$$H_B\left(\alpha_3;\gamma_3;z_1,z_2,z_3\right)=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}.$$

Thus

$$\mathrm{D}H_B\left(\alpha_3;\gamma_3;z_1,z_2,z_3\right)$$
$$=\mathrm{D}\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$
$$=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(n_1+n_2+n_3)(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$
$$=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_3)_{n_3}}(z_1)^{n_1+1}(z_2)^{n_2}(z_3)^{n_3}$$
$$+\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_3)_{n_2+1+n_3}}{n_1!n_2!n_3!(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2+1}(z_3)^{n_3}$$
$$+\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_3)_{n_2+n_3+1}}{n_1!n_2!n_3!(\gamma_3)_{n_3+1}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3+1}$$
$$=(z_1)\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$
$$+(z_2)\sum_{n_1,n_2,n_3=0}^{\infty}(\alpha_3+n_2+n_3)\frac{(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$
$$+(z_3)\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_3+n_2+n_3)}{(\gamma_3+n_3)}\frac{(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}.$$

The function

(3.7) $$H_B\left(\alpha_3;\gamma_3;z_1,z_2,z_3\right)=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}.$$

has a solution of the equation

(3.8) $$\left[(D-z_1)H_B-\alpha_3 z_2 H_B(\alpha_3+1;\gamma_3;z_1,z_2,z_3)-\frac{\alpha_3 z_3}{\gamma_3}H_B(\alpha_3+1;\gamma_3+1;z_1,z_2,z_3)\right]=0$$

2- The function:

$$H_B\left(-,-,-;z_1,z_2,z_3\right)=\sum_{n_1,n_2,n_3=0}^{\infty}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}.$$

Thus

$$\mathrm{D}H_B(-,-,-;z_1,z_2,z_3)=\sum_{n_1,n_2,n_3=0}^{\infty}(n_1+n_2+n_3)(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

$$=\sum_{n_1,n_2,n_3=0}^{\infty}(z_1)^{n_1+1}(z_2)^{n_2}(z_3)^{n_3}+\sum_{n_1,n_2,n_3=0}^{\infty}z_1)^{n_1}(z_2)^{n_2+1}(z_3)^{n_3}+\sum_{n_1,n_2,n_3=0}^{\infty}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3+1}$$

$$\mathrm{D}H_B(-,-,-;z_1,z_2,z_3)=(z_1+z_2+z_3)\sum_{n_1,n_2,n_3=0}^{\infty}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}.$$

The function

$$H_B(-,-,-;z_1,z_2,z_3)=\sum_{n_1,n_2,n_3=0}^{\infty}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}.$$

has a solution of the equation

$$[D-(z_1+z_2+z_3)]H_B(-,-,-;z_1,z_2,z_3)=0$$

Now will given a numerical example for one of these differential equations that we get, by giving a numerical value for constant numbers in any previous equation as the one used in (3.8) and we get an equation representing the surface sphere equation, its solution is solution for the equation (3.8) after substituting the same numerical value. This clarifies the idea of the study.

Example *:*
Here we shall take the equation in (3.8) as follows:

$$(2.3)\qquad\left[(D-z_1)H_B-\alpha_3 z_2 H_B(\alpha_3+1;\gamma_3;z_1,z_2,z_3)-\frac{\alpha_3 z_3}{\gamma_3}H_B(\alpha_3+1;\gamma_3+1;z_1,z_2,z_3)\right]=0$$

which has a general solution in equation (3.8) we can consider that

$$(3.10)\qquad\begin{cases}\alpha_3=\gamma_3=1\\ n_1=n_2=n_3=1\end{cases}$$

Substituting from (3.10) in the equation (3.8) we get :-

$$H_B(1;1;z_1,z_2,z_3)=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(1)_{n_2+n_3}}{1!1!1!(1)_{n_3}}(z_1)(z_2)(z_3)\text{ and}$$

since

$$n_1=n_2=n_3=1\ ,\ (1)_{n_2+n_3}=1(2)(3)...(n_2+n_3-1)!\text{ and }(1)_{n_3}=1(2)(3)\ldots(n_3-1)!.$$

$$\left[(D-z_1)H_B(1;1;z_1,z_2,z_3)-z_2H_B(1;1;z_1,z_2,z_3)-z_3H_B(1;1;z_1,z_2,z_3)\right]=0$$

$$\left[D-(z_1+z_2+z_3)\right]H_B(1;1;z_1,z_2,z_3)=0$$

Applying that and the partial differential equation (3.9) we see that: -

$$[D-(z_1+z_2+z_3)H_A+0]=0$$

$$[D-(z_1+z_2+z_3)](z_1z_2z_3)=0$$

$$D(z_1z_2z_3)-(z_1+z_2+z_3)(z_1z_2z_3)=(z_1z_2z_3)\left[3-(z_1+z_2+z\ \right]=0$$

i.e. $\qquad (z_1+z_2+z_3)-3=0$

which is a surface equation in hypersphere has a solution in the form:

$$H_B(1;1;z_1,z_2,z_3)=z_1z_2z_3$$

IV. Integral Representations For Certain H_B -Hypergeometric Function Of Three Variables Due To Lauricella And Srivastava

Now we consider H_B - hypergeometric function as follows

$$H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;z_1,z_2,z_3) = \sum_{n_1,n_2,n_3=0}^{\infty} \frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(z_1)^{n_1}(z_2)^{n_2}(z_3)^{n_3}$$

and

(4.1) $$H_B[\alpha+\beta+1,\gamma+\delta+1,\lambda+\mu+1;\alpha+1,\delta+1,\beta+1;z_1,z_2,z_3]$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\delta+1)\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\alpha+\beta+1)\Gamma(\gamma+\delta+1)\Gamma(\lambda+\mu+1)}\frac{2^{\alpha+\beta+\gamma+\delta+\lambda+\mu}}{\pi^3}$$

$$.\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta)i\theta+(\gamma-\delta)i\varphi+(\lambda-\mu)i\psi}\cos^{\alpha+\beta}\theta\cos^{\gamma+\delta}\varphi\cos^{\lambda+\mu}\psi$$

$$.\left[1-(1+\cos 2\theta - i\sin 2\theta)(1+\cos 2\varphi + i\sin 2\varphi)x\right]^{-\gamma-1}$$

$$.\left[1-(1+\cos 2\varphi - i\sin 2\varphi)(1+\cos 2\psi + i\sin 2\psi)y\right]^{-\lambda-1}$$

$$.\left[1-(1+\cos 2\theta - i\sin 2\theta)(1+\cos 2\psi + i\sin 2\psi)z\right]^{-\mu-1} d\theta d\varphi d\psi$$

where

$$\mathrm{Re}(\alpha+\beta)>-1,\ \mathrm{Re}(\gamma+\delta)>-1 \ and\ \ \mathrm{Re}(\lambda+\mu)>-1.$$

Proof :

$$H_B[\alpha+\beta+1,\gamma+\delta+1,\lambda+\mu+1;\alpha+1,\delta+1,\beta+1;z_1,z_2,z_3]$$

$$= \sum_{m,n,p=0}^{\infty}\frac{(\alpha+\beta+1,m+n)(\gamma+\delta+1,p+n)(\lambda+\mu+1,m+p)}{m!n!p!(\alpha+1,m)(\delta+1,n)(\beta+1,p)}z_1^m z_2^n z_3^p$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\delta+1)\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\alpha+\beta+1)\Gamma(\gamma+\delta+1)\Gamma(\lambda+\mu+1)}$$

$$.\sum_{m,n,p=0}^{\infty}\frac{(\lambda+1,m)(\delta+1,n)(\mu+1,p)}{m!n!p!}z_1^m z_2^n z_3^p$$

$$.\frac{\Gamma(\alpha+\beta+m+p+1)\Gamma(\gamma+p+1)}{\Gamma(\alpha+m+1)\Gamma(\beta+p+1)}\frac{\Gamma(\beta+p+1)(\gamma+\delta+m+n+1)}{\Gamma(\delta+n+1)\Gamma(\gamma+m+1)}$$

$$.\frac{(\lambda+\mu+n+p+1)}{\Gamma(\lambda+n+1)\Gamma(\mu+p+1)}\frac{(\gamma+m+1)\Gamma(\lambda+n+1)\Gamma(\mu+p+1)}{1}$$

Now, it is known that (Whittaker and Watsin [ ])

$$\frac{\Gamma(\lambda+n+1)}{\Gamma(\lambda+1)\Gamma(\mu+1)} = \frac{2^{\lambda+\mu}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(\lambda-\mu)\theta}\cos^{\lambda+\mu}\theta\, d\theta$$

where $\mathrm{Re}(\lambda+\mu)>-1$

so that the right hand side of (4.1) becomes

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\delta+1)\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\alpha+\beta+1)\Gamma(\gamma+\delta+1)\Gamma(\lambda+\mu+1)}\frac{2^{\alpha+\beta+\gamma+\delta+\lambda+\mu}}{\pi^3}$$

$$. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta)i\theta+(\gamma-\delta)i\varphi+(\lambda-\mu)i\psi} \cos^{\alpha+\beta}\theta \cos^{\gamma+\delta}\varphi \cos^{\lambda+\mu}\psi$$

$$. \left[1-\left(1+\cos 2\theta - i \sin 2\theta\right)\left(1+\cos 2\varphi + i \sin 2\varphi\right) z_1\right]^{-\gamma-1}$$

$$. \left[1-\left(1+\cos 2\varphi - i \sin 2\varphi\right)\left(1+\cos 2\psi + i \sin 2\psi\right) z_2\right]^{-\lambda-1}$$

$$. \left[1-\left(1+\cos 2\theta - i \sin 2\theta\right)\left(1+\cos 2\psi + i \sin 2\psi\right) z_3\right]^{-\mu-1} d\theta d\varphi d\psi$$

Using the identities

$$2e^{i\theta}\cos\theta = 2\left(\cos^2\theta + i \sin\theta\cos\theta\right)$$

$$= 1+\cos 2\theta + i \sin 2\theta$$

and

$$2e^{-i\theta}\cos\theta = 1+\cos 2\theta - i \sin 2\theta$$

Then we have

$$H_B\left[\alpha+\beta+1, \gamma+\delta+1, \lambda+\mu+1; \alpha+1, \delta+1, \beta+1; x, y, z\right] =$$

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\delta+1)\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\alpha+\beta+1)\Gamma(\gamma+\delta+1)\Gamma(\lambda+\mu+1)} \frac{2^{\alpha+\beta+\gamma+\delta+\lambda+\mu}}{\pi^3}$$

$$. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{(\alpha-\beta)i\theta+(\gamma-\delta)i\varphi+(\lambda-\mu)i\psi} \cos^{\alpha+\beta}\theta \cos^{\gamma+\delta}\varphi \cos^{\lambda+\mu}\psi$$

$$. \left[1-\left(1+\cos 2\theta - i \sin 2\theta\right)\left(1+\cos 2\varphi + i \sin 2\varphi\right) z_1\right]^{-\gamma-1}$$

$$. \left[1-\left(1+\cos 2\varphi - i \sin 2\varphi\right)\left(1+\cos 2\psi + i \sin 2\psi\right) z_2\right]^{-\lambda-1}$$

$$. \left[1-\left(1+\cos 2\theta - i \sin 2\theta\right)\left(1+\cos 2\psi + i \sin 2\psi\right) z_3\right]^{-\mu-1} d\theta d\varphi d\psi$$

References

[1]. J.L. Burchnall, T.W. Chaundy, Expansions Of Appell's Double Hypergeometric Functions, Quart. J. Math. Oxford Ser. 11 (1940) 249–270.

[2]. J.L. Burchnall, T.W. Chaundy, Expansions Of Appell's Double Hypergeometric Functions. II, Quart. J. Math. Oxford Ser. 12 (1941) 112–128.

[3]. T.W. Chaundy, Expansions Of Hypergeometric Functions, Quart. J. Math. Oxford Ser. 13 (1942) 159–171.

[4]. Chaundy, T.W. "On Appell's Fourth Hypergeometric Functions". The Quart. J. Mathematical Oxford (2) 17 (1966) Pp.81-85.

[5]. F.I. Frankl, Selected Works In Gas Dynamics, Nauka, Moscow, 1973 (In Russian).

[6]. A. Hasanov, On A Mixed Problem For The Equation Sign Y|Y|Muxx + Xnuyy = 0, Izv. Akad. Nauk Uzssr Ser. Fiz.- Mat. Nauk 2 (1982) 28–32 And 76 (In Russian).

[7]. JOSHI, C.M. And Bissu S.K. "Some Inequalities Of Hypergeometric Function Of Three Variables". Jnanabha, Vol. 21 (1991) Pp.151-164.

[8]. G. Lauricella, Sulle Funzioni Ipergeometriche A Più Variabili, Rend. Circ. Mat. Palermo 7 (1893) 111–158.

[9]. G. Lohöfer, Theory Of An Electromagnetically Deviated Metal Sphere. I: Absorbed Power, SIAM J. Appl. Math. 49 (1989) 567–581.

[10]. A.W. Niukkanen, Generalised Hypergeometric Series NF(X1, . . . , Xn) Arising In Physical And Quantum Chemical Applications, J. Phys. A: Math. Gen. 16 (1983) 1813–1825.

[11]. S.B. Opps, N. Saad, H.M. Srivastava, Some Reduction And Transformation Formulas For The Appell Hypergeometric Function F2, J. Math. Anal. Appl. 302 (2005) 180–195.

[12]. P.A. Padmanabham, H.M. Srivastava, Summation Formulas Associated With The Lauricella Function F(R) A , Appl. Math. Lett. 13 (1) (2000) 65–70.

[13]. Raniville, Earld "Special Functions". New York (1960).

[14]. H.M. Srivastava, Hypergeometric Functions Of Three Variables, Gan.Ita 15 (1964) 97–108.

[15]. H.M. Srivastava, Some Integrals Representing Triple Hypergeometric Functions, Rend. Circ. Mat. Palermo (Ser. 2) 16 (1967) 99–115.

[16]. H.M. Srivastava, A Class Of Generalised Multiple Hypergeometric Series Arising In Physical And Quantum Chemical Applications, J. Phys. A: Math. Gen. 18 (1985) L227–L234.

[17]. H.M. Srivastava, P.W. Karlsson, Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York, Chichester, Brisbane And Toronto, 1985.

[18]. H.M. Srivastava, H.L.Manocha, A Treatise On Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester),Wiley, New York, Chichester, Brisbane And Toronto, 1984.