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Integral Formulae involving Aleph Function

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Abstract
In this paper, we have established various integral formulas involving product of Aleph function with exponential function and Gauss's hypergeometric function. The integral formulae evaluated here are of general in nature and may be viewed as Euler transform of the Aleph function multiplied with exponential function and Gauss hypergeometric function. Several new and known results can be obtained as sepcial cases.

Key words : *Aleph (ℵ)-function, I-function, H-function, Gauss hypergeometric function, Mellin-Barnes type contour integral.*

I. Introduction

The Aleph function which is a general higher transcendental function was introduced by Südland [10,11]. The notation and complete definition in terms of Mellin-Barnes type integral is as follows :

$$\aleph[z] = \aleph^{m,n}_{p_i,q_i,\tau_i;r}[z] = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z \left| \begin{matrix} (a_j,A_j)_{1,n}, [\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (b_j,B_j)_{1,m}, [\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) z^{-s} ds \qquad (1.1)$$

for all z ≠ 0, where $\omega = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s).\prod_{j=1}^{n}\Gamma(1-a_j-A_j s)}{\sum_{i=1}^{r}\tau_i \prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s).\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-B_{ji}s)} \qquad (1.2)$$

Here, L is a Barnes path of integration. The integration path $L = L_{\omega\gamma\infty}$, $\gamma \in R$, starts at $\gamma - \omega\infty$ and runs to $\gamma + \omega\infty$ in the s-plane, curving if necessary in such a way that the poles of the gamma functions $\Gamma(1-a_j-A_j s)$, j = 1, 2…n, do not coincides with the poles of the gamma functions $\Gamma(b_j+B_j s)$, $j = 1,...,m$. All poles of (1.2) are assumed to be simple and the empty product is taken as unity. For i = 1,...,r, the parameters τ_i are positive integers and the parameters p_i, q_i are non-negative integers with $0 \le n \le p_i, 0 \le m \le q_i$. The parameters a_j, b_j, a_{ji}, b_{ji} are complex and A_j, B_j, A_{ji}, B_{ji} are positive numbers. The integral (1.1) exist if the following conditions are satisfied :

$$\phi_i > 0, \ |arg(z)| < \frac{\pi}{2}\phi_i \ ; \ i = 1,.........,r \ , \qquad (1.3)$$

$$\phi_i \ge 0, \ |arg(z)| < \frac{\pi}{2}\phi_i \ ; \ \text{Re}\{\psi_i\} + 1 < 0 \ , \qquad (1.4)$$

where

$$\phi_i = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_i \left(\sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=m+1}^{q_i} B_{ji} \right), \qquad (1.5)$$

$$\psi_i = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_i \left(\sum_{j=m+1}^{q_i} b_{ji} - \sum_{j=n+1}^{p_i} a_{ji} \right) + \frac{1}{2}(p_i - q_i); i = 1, ...r , \qquad (1.6)$$

By taking $\tau_i = 1, (i = 1, ..., r)$, the Aleph (ℵ)- function (1.1) reduces to I-function [7].

$$I_{p_i,q_i;r}^{m,n}[z] = \aleph_{p_i,q_i,1;r}^{m,n}\left[z \left| \begin{matrix} (a_j, A_j)_{1,n}, ..., (a_{ji}, A_{ji})_{n+1,p_i} \\ (b_j, B_j)_{1,m}, ..., (b_{ji}, B_{ji})_{m+1,q_i} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi\omega} \int_L \Omega_{p_i,q_i,1;r}^{m,n}(s) z^{-s} ds , \qquad (1.7)$$

where the existence conditions are given by (1.3)-(1.6) and $\Omega_{p_i,q_i,1;r}^{m,n}(s)$ is given by (1.2).

For r = 1 and $\tau_1 = 1$, we get Aleph function in the form of Fox's H-function [2] :

$$H_{p,q}^{m,n}[z] = \aleph_{p_i,q_i,1;1}^{m,n}\left[z \left| \begin{matrix} (a_j, A_j) \\ (b_j, B_j) \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi\omega} \int_L \Omega_{p_i,q_i,1;1}^{m,n}(s) z^{-s} ds \qquad (1.8)$$

where the kernel $\Omega_{p_i,q_i,1;1}^{m,n}$ is defined by (1.2)

The H-function is a generalization of Meijer's G-function which is further generalization of hypergeometric function ${}_pF_q$.

The hypergeometric function ${}_2F_1$ is defined as [4, p 45, chap 4]:

$${}_2F_1(a, b ; c ; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad (1.9)$$

where c is neither zero nor a negative integer and $(a)_n, (b)_n, (c)_n$ are Pochhammer symbol [4, p22, sec 18] defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)......(a+n-1),\ n \in \mathbf{N}$$

and $(a)_0 = 1$, $a \neq 0$

For detailed study of these Special functions, refer to [3, 6, 8, 9].

II. Main Results

In this section, we derive integrals involving product of the Aleph function with exponential function and Gauss hypergeometric function.

First Integral :

$$I_1 \equiv \int_0^t x^{\rho-1} (t-x)^{\sigma-1} e^{-xy} {}_2F_1\left[\alpha, \beta, \gamma; cx^{\xi}(t-x)^{\eta}\right]$$

$$\times \aleph_{p_i,q_i,\tau_i;r}\left[zx^{\mu}(t-x)^{\nu} \left| \begin{matrix} (a_j, A_j)_{i,n}, [\tau_i(a_{ji}, A_{ji})]_{n+1,p_i;r} \\ (b_{j,} B_j)_{1,m}, [\tau_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix} \right. \right] dx$$

$$= e^{-yt} t^{\rho+\sigma-1} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{c^k}{k!} \frac{y^{n-k}}{(n-k)!} t^{(\xi+\eta-1)k+n}$$

$$\times \aleph^{m,n+2}_{p_i+2,q_i+1,\tau_i;r} \left[zt^{\mu+v} \left| \begin{matrix} (1-\rho-\xi k,\mu),(1-\sigma-(\eta-1)k-n,v), \\ (b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r}, \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (a_j,A_j)_{1,n},\ [\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (1-\rho-\sigma-(\xi+\eta-1)k-n,\mu+v) \end{matrix} \right] \tag{2.1}$$

provided the following existence conditions are satisfied :

(1) ξ and η are non-negative integers such that $\xi + \eta \geq 1$,

(2) $\mu \geq 0$, $v \geq 0$ (not both zero simultaneously),

(3) $\phi_i > 0,\ \psi_i < 0;\ |\arg z| < \frac{1}{2}\phi_i \pi,\ \forall\ i = 1, ...r$

where

$$\phi_i = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_i \left(\sum_{j=n+1}^{p_i} A_{ji} + \sum_{j=m+1}^{q_i} B_{ji} \right),$$

$$\psi_i = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_i \left(\sum_{j=m+1}^{q_i} b_{ji} - \sum_{j=n+1}^{p_i} a_{ji} \right) + \frac{1}{2}(p_i - q_i),$$

(4) $\mathrm{Re}(\rho) + \mu \min_{1 \leq j \leq n} \left[\mathrm{Re}(b_j / B_j) \right] > 0,$

$\mathrm{Re}(\rho) + v \min_{1 \leq j \leq n} \left[\mathrm{Re}(b_j / B_j) \right] > 0.$

Proof :

$$I_1 = \int_0^t x^{\rho-1} (t-x)^{\sigma-1} e^{-yt} e^{(t-x)y} {}_2F_1\left[\alpha, \beta; \gamma; c x^{\xi} (t-x)^{\eta} \right]$$

$$\times \aleph^{m,n}_{p_i,q_i,\tau_i;r} \left[zx^{\mu} (t-x)^{v} \left| \begin{matrix} (a_j,A_j)_{1,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r} \\ (b_j,B_j)_{1,m},[\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r} \end{matrix} \right. \right] dx \tag{2.2}$$

we know that

$$e^{(t-x)y} = \sum_{n=0}^{\infty} \frac{(t-x)^n y^n}{n!} \tag{2.3}$$

using (2.3), (1.1) and (1.9) in (2.2) we have

$$I_1 = e^{-yt} \int_0^t x^{\rho-1} (t-x)^{\sigma-1} \sum_{n=0}^{\infty} \frac{(t-x)^n y^n}{n!} \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{c^k x^{\xi k} (t-x)^{\eta k}}{k!}$$

$$\times \left\{ \frac{1}{2\pi\omega} \int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) z^{-s} x^{-\mu s} (t-x)^{-vs} ds \right\} dx$$

$$= e^{-yt} \int_0^t x^{\rho-1} (t-x)^{\sigma-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{c^k x^{\xi k} (t-x)^{\eta k+n}}{k!} \frac{y^n}{n!}$$

$$\times \left\{ \frac{1}{2\pi\omega} \int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) z^{-s} x^{-\mu s} (t-x)^{-vs} ds \right\} dx \tag{2.4}$$

Using the following identity [4 , p 56, Lemma 10]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k) \tag{2.5}$$

in (2.4) we get,

$$I_1=e^{-yt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\;\frac{c^k x^{\xi k}(t-x)^{\eta k+n-k}}{k!}\;\frac{y^{n-k}}{(n-k)!}$$

$$\times\left\{\frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)\,z^{-s}x^{-\mu s}(t-x)^{vs}\,ds\right\}dx \tag{2.6}$$

On interchanging the order of integration and summation which is justified under the conditions given with the result (2.1), we obtain

$$I_1=e^{-yt}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\;\frac{c^k}{k!}\;\frac{y^{n-k}}{(n-k)!}\;\frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)\,z^{-s}$$

$$\times\left\{\int_0^t x^{\rho+\xi k-\mu s-1}(t-x)^{\sigma+(\eta-1)k+n-vs-1}\,dx\right\}ds \tag{2.7}$$

Evaluating the inner integral in (2.7) by means of the beta function B (p, q) formula

$$\int_0^t x^{p-1}(t-x)^{q-1}\,dx=t^{p+q-1}\mathbf{B}(p,q)\,;\quad \min\left(\mathrm{Re}(p),\mathrm{Re}(q)\right)>0, \tag{2.8}$$

we observe that

$$I_1=e^{-yt}t^{\rho+\sigma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\;\frac{c^k}{k!}\;\frac{y^{n-k}}{(n-k)!}\;t^{(\xi+\eta-1)k+n}$$

$$\times\frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i\tau_i;r}(s).\;\frac{\Gamma(\rho+\xi k-\mu s)\Gamma\left(\sigma+(\eta-1)k+n-vs\right)}{\Gamma\left(\rho+\sigma+(\xi+\eta-1)k+n-(\mu+v)s\right)}\,z^{-s}t^{-(\mu+v)s}ds$$

Using (1.2) and interpreting the contour integral by (1.1), we get

$$I_1=e^{-yt}t^{\rho+\sigma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\;\frac{c^k}{k!}\;\frac{y^{n-k}}{(n-k)!}\;t^{(\xi+\eta-1)k+n}$$

$$\times\aleph^{m,\,n+2}_{p_i+2,q_i+1,\tau_i;r}\left[zt^{\mu+v}\left|\begin{matrix}(1-\rho-\xi k,\mu),(1-\sigma-(\eta-1)k-n,v),(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\left[\tau_i(b_{ji},B_{ji})\right]_{m+1,q_i;r},(1-\rho-\sigma-(\xi+\eta-1)k-n,\mu+v)\end{matrix}\right.\right].$$

Corollary :

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xy}{}_2F_1\left[\alpha,\beta;\gamma;cx^{\xi}(t-x)^{\eta}\right]$$

$$\times\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[zx^{-\mu}(t-x)^{-v}\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i;r}\\(b_j,B_j)_{1,m},\left[\tau_i(b_{ji},B_{ji})\right]_{m+1,q_i;r}\end{matrix}\right.\right]dx$$

$$=e^{-yt}t^{\rho+\sigma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\;\frac{c^k}{k!}\;\frac{y^{n-k}}{(n-k)!}\;t^{(\xi+\eta-1)k+n}$$

$$\times\aleph^{m+2,n}_{p_i+1,q_i+2,\tau_i;r}\left[zt^{-\mu-v}\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right)_{n+1,p_i;r},\\(\rho+\xi k,\mu),(\sigma+(\eta-1)k+n,v),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(\rho+\sigma+(\xi+\eta-1)k+n,\mu+v\right)\\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r}\end{matrix}\right], \tag{2.9}$$

provided the following conditions and the conditions (1),(3) given with (2.1) are satisfied :

$$\operatorname{Re}(\rho)-\mu\max_{1\le j\le n}\left[\operatorname{Re}\left(\left(a_j-1\right)/A_j\right)\right]>0,$$

$$\operatorname{Re}(\rho)-v\max_{1\le j\le n}\left[\operatorname{Re}\left(\left(a_j-1\right)/A_j\right)\right]>0.$$

Second Integral :

$$I_2\equiv\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xy}{}_2F_1\left[\alpha,\beta;\gamma;cx^{\xi}(t-x)^{\eta}\right]$$

$$\times\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[zx^{-\mu}(t-x)^{v}\left|\begin{matrix}\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r}\\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r}\end{matrix}\right.\right]dx$$

$$=e^{-yt}t^{\rho+\sigma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{c^k}{k!}\frac{y^{n-k}}{(n-k)!}t^{(\xi+\eta-1)k+n}$$

$$\times\aleph_{p_i+1,q_i+2,\tau_i;r}^{m+1,n+1}\left[zt^{-\mu+v}\left|\begin{matrix}\left(1-\sigma-(\eta-1)k-n,v\right),\left(a_j,A_j\right)_{1,n},\\ \left(\rho+\xi k,\mu\right),\left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r}\\ \left(1-\rho-\sigma-(\xi+\eta-1)k-n,v-\mu\right)\end{matrix}\right], \tag{2.10}$$

provided the conditions (1), (3) given with (2.1) and the following conditions are satisfied :

$\mu\ge 0$, $v>0$ such that $v-\mu\ge 0$,

$$\operatorname{Re}(\rho)-\mu\max_{1\le j\le n}\left[\operatorname{Re}\left(\left(a_j-1\right)/A_j\right)\right]>0,$$

$$\operatorname{Re}(\sigma)+v\min_{1\le j\le n}\left[\operatorname{Re}\left(b_j/B_j\right)\right]>0.$$

Proof : The integral (2.10) can be easily evaluated using the procedure discussed in (2.1).

Corollary :

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xy}{}_2F_1\left[\alpha,\beta;\gamma;cx^{\xi}(t-x)^{\eta}\right]$$

$$\times\aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[zx^{-\mu}(t-x)^{v}\left|\begin{matrix}\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r}\\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r}\end{matrix}\right.\right]dx$$

$$=e^{-yt}t^{\rho+\sigma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{c^k}{k!}\frac{y^{n-k}}{(n-k)!}t^{(\xi+\eta-1)k+n}$$

$$\times\aleph_{p_i+2,q_i+1,\tau_i;r}^{m+1,n+1}\left[zt^{-\mu+v}\left|\begin{matrix}\left(1-\sigma-(\eta-1)k-n,v\right),\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r},\\ \left(\rho+\xi k,\mu\right),\left(b_j,B_j\right)_{1,m},\end{matrix}\right.\right.$$

$$\left.\begin{matrix} \rho+\sigma+(\xi+\eta-1)k+n,\mu-v \\ \left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r} \end{matrix}\right], \qquad (2.11)$$

provided the following conditions along with the conditions (1), (3) given with (2.1) are satisfied :

$\mu > 0$, $\nu \geq 0$ such that $\mu - \nu \geq 0$,

$$\mathrm{Re}(\rho)+\mu \min_{1\leq j\leq m}\left[\mathrm{Re}\left(b_j/B_j\right)\right]>0,$$

$$\mathrm{Re}(\sigma)-v \max_{1\leq j\leq n}\left[\mathrm{Re}\left(\left(a_j-1\right)/A_j\right)\right]>0.$$

Third Integral :

$$I_3 \equiv \int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xy}{}_2F_1\left[\alpha,\beta;\gamma;cx^{\xi}(t-x)^{\eta}\right]$$

$$\times \aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[zx^{\mu}(t-x)^{-v}\left|\begin{matrix}\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r} \\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r}\end{matrix}\right.\right]dx$$

$$=e^{-yt}t^{\rho+\sigma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{c^k}{k!}\frac{y^{n-k}}{(n-k)!}\ t^{(\xi+\eta-1)k+n}$$

$$\times \aleph_{p_i+2,\,q_i+1,\tau_i;r}^{m+1,n+1}\left[zt^{\mu-v}\left|\begin{matrix}(1-\rho-\xi k,\mu),\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r}, \\ (\sigma+(\eta-1)k+n,v),\left(b_j,B_j\right)_{1,m},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(\rho+\sigma+(\xi+\eta-1)k+n,v-\mu) \\ \left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r}\end{matrix}\right], \qquad (2.12)$$

provided the following conditions along with the conditions (1), (3) given with (2.1) are satisfied :

$\mu \geq 0$, $\nu > 0$ such that $\nu - \mu \geq 0$,

$$\mathrm{Re}(\rho)-\mu \max_{1\leq j\leq n}\left[\mathrm{Re}\left(a_j-1\right)/A_j\right]>0,$$

$$\mathrm{Re}(\sigma)+v \min_{1\leq j\leq m}\left[\mathrm{Re}\left(b_j/B_j\right)\right]>0.$$

Proof : It can be easily proved using the procedure as discussed in (2.1)

Corollary :

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-xy}{}_2F_1\left[\alpha,\beta;\gamma;cx^{\xi}(t-x)^{\eta}\right]$$

$$\times \aleph_{p_i,q_i,\tau_i;r}^{m,n}\left[zx^{\mu}(t-x)^{-v}\left|\begin{matrix}\left(a_j,A_j\right)_{1,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r} \\ \left(b_j,B_j\right)_{1,m},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{m+1,q_i;r}\end{matrix}\right.\right]dx$$

$$=e^{-yt}t^{\rho+\sigma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{c^k}{k!}\frac{y^{n-k}}{(n-k)!}\ t^{(\xi+\eta-1)k+n}$$

$$\times \aleph^{m+1,n+1}_{p_i+1,q_i+2,\tau_i;r}\left[zt^{\mu-\nu} \left| \begin{matrix} (1-\sigma-\xi k,\mu),(a_j,A_j)_{1,n}, \\ (\sigma+(\eta-1)k+n,\nu),(b_j,B_j)_{1,m}, \end{matrix} \right. \begin{matrix} [\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r} \\ [\tau_i(b_{ji},B_{ji})]_{m+1,q_i;r},(1-\rho-\sigma-(\xi+\eta-1)k-n,\mu-\nu) \end{matrix} \right], \qquad (2.13)$$

provided the conditions (1), (3) given in (2.1) along with the following conditions are satisfied :

$\mu > 0$, $\nu \geq 0$ such that $\mu - \nu \geq 0$,

$$\mathrm{Re}(\rho)+\mu \min_{1\leq j\leq m}\left[\mathrm{Re}\left(b_j/B_j\right)\right]>0,$$

$$\mathrm{Re}(\sigma)-\nu \max_{1\leq j\leq n}\left[\mathrm{Re}\left(a_j-1\right)/A_j\right]>0.$$

III. Special Cases

By suitable substitution and adjustment of the parameters in the Integral formulae evaluated in this paper, we can easily obtain several new and known integrals involving special functions such as I-function, H-function, Meijers G-Function, etc. [3, 8]

(i) For $\tau_1 = ... = \tau_r = 1$, as the Aleph ($\aleph$)-function reduces to I-function, we can obtain results given by Saha et al. [5].

(ii)Taking r = 1 , $\tau_1 = 1, t = 1, \eta = 0$, the results given in [1] can be obtained.

We can also obtain a large number of particular cases of our results as the Euler transform of the Aleph function multiplied with special functions.

References

[1]. L.K. Arora and U.K. Saha, Integrals involving hypergeometric function and H-function, J. Indian Acad. Math., 32 (1) (2010), 243-429.

[2]. C. Fox, the G. and H-functions as symmetrical Fourier kernels, Trans. Amer. Math. Soc., 98 (1961), 395-429.

[3]. A.M. Mathai and R.K. Saxena, H.J. Haubold, The H-function : Theory and Applications, Springer, New York (2010).

[4]. E.D. Rainville, Special Functions, The Macmillan Company, New York, (1960).

[5]. U.K. Saha, L.K. Arora and B.K. Dutta, Integrals involving I-function, Gen. Math. Notes, 6(1) (2011), 1-14.

[6]. R.K. Saxena and T.K. Pogany, On fractional integration formulae for Aleph functions, Appl. Math. Comput. 218 (2011), 985-990.

[7]. V.P. Saxena, Formal solution of certain new pair of dual integral equations involving H-functions, Proc.Nat. Acad. Sci., India, Sec A, 51 (1982), 366-375.

[8]. V.P. Saxena. The I-Funtion, Anamaya Publishers, New Delhi, (2008).

[9]. H.M. Srivastava, K.C. Gupta and S.P. Goyal. The H-Functions of One and Two Variables with Applications, South Asian Publishers, New Delhi, Madras, (1982).

[10]. N. Südland, B. Baumann and T.F. Nonnenmacher, Open Problem : who knows about the Aleph ($\aleph$) –function?, Fract. Calc. Appl. Anal. 1 (4) (1998), 401-402.

[11]. N. Südland,B. Baumann and T.F. Nonnenmacher, Fractional driftless Fokker-Planck equation with power law diffusion coefficients, in V.G. Gangha, E.W. Mayr, W.G. Vorozhtsov (Eds.), Computer Algebra in Scientific computing (CASS Konstanz 2001), Springer, Berlin, (2001), 513-525.