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# Oscillations of All Solutions of Functional Difference Inequalities

Pon Sundar[1] and K. Revathi[2]

[1] *Om Muruga College of Arts and Science, Mettur (Tk), Salem - 636 303, Tamil Nadu, India.*

[2] *Department of Mathematics, Sri Sarada Niketan College of Arts and Science, Salem - 636 354, Tamil Nadu, India.*

***Abstract:*** *In this paper, we consider the oscillatory behavior of the functional difference inequality of the form*

$$(-1)^z \Delta^m x(n) sgnx\ (n) \geq p(n) \prod_{i=1}^{k} \left| x(g_i(n)) \right|^{\alpha_i} \qquad (E_z)$$

$m \geq 2$ , $z$ *is a natural number,* $n \in N(n_0) = \{n_0, n_0 + 1, \cdots\}$ *and* $\alpha_i \in \mathrm{P}_+ = [0, \infty)$ *with* $\alpha_1 + \cdots + \alpha_k = 1$ *. The sequence* $p(n)$ *is not identically zero and the sequence* $g_i : \mathrm{P}_+ \to \mathrm{P}_+$ *are such that* $\lim_{n \to \infty} g_i(n) = \infty$ *. The propose of this paper is to study oscillations of solutions of inequality* $(E_z)$ *generated by general deviating arguments* $g_i$ *(not necessarily delay or advanced arguments). Some specific comparison to known results will be made in the text of the paper.*

***Keywords:*** *Oscillatory behavior, general deviating arguments, monotonicity, specific comparison, functional difference inequalities.*
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## I. Introduction

In this paper, we are considered with the oscillatory behavior of the functional difference inequality of the form

$$(-1)^z \Delta^m x(n) sgnx\ (n) \geq p(n) \prod_{i=1}^{k} \left| x(g_i(n)) \right|^{\alpha_i} \qquad (E_z)$$

where $z$ is a natural number, $m \geq 2$ and $\Delta$ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$ and $\Delta^i x(n) = \Delta(\Delta^{i-1} x(n)),\ \ i = 1,2, \cdots$ .

We assume the following conditions on the inequality $(E_z)$

1. $\alpha_i \in \mathrm{P}_+ = [0, \infty)$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$ .
2. The real sequence $p(n)$ is not identically zero in every neighborhood of infinity.
3. The sequence of integers $g_i : \mathrm{P}_+ \to \mathrm{P}_+\ \ (i = 1,2, \cdots, k)$ are such that $\lim_{n \to \infty} g_i(n) = \infty$ .

For all $r \in \mathrm{P} = (-\infty, \infty)$ and $s$ a nonnegative integer, the factorial expression is defined as

$$(r)^{(s)} = \prod_{i=0}^{s-1} (r - i) \quad with \quad (r)^{(0)} = 1.$$

By a solution of inequality $(E_z)$ we mean a real sequence $\{x(n)\}$ which is defined for all $n \in N(n_0)$ and satisfies the inequality $(E_z)$ for all sufficiently large $n \in N(n_0)$. Our attention is restricted to those solutions which are nontrivial in the seme that $\sup\{|x(n)| : n \geq N\} > 0$ for any $N > n_0$ . Such a solution is said to be oscillatory, if it is neither eventually positive nor eventually negative and non-oscillatory otherwise.

The aim of this paper is to study oscillatory of all solutions of $(E_z)$ generated by general deviating arguments (not necessity delay or advanced arguments). The main results of this paper are new and are independent of the

analogous ones known for delay and advanced difference equations. Some specific comparisons to known results will be made in the text of the paper.
The problem of oscillation and non-oscillation of solutions of difference equations / inequalities has received a considerable attention during the last few years. Among numerous papers dealing with the subject, we refer in particular [2, 3, 5-13, 15-23] and references citied therein. However, it seems that very little work has been done on the oscillating behavior of difference inequalities.

The following notations will be used throughout this paper.

$$D = \{n \in P_+ : g_i(n) \le n, i = 1,2, \cdots, k\}$$

$$A = \{n \in P_+ : g_i(n) \ge n, i = 1,2, \cdots, k\}$$

Let $g_i^*, d_i, a_i : P_+ \to P_+ \quad (i = 1,2, \cdots, k)$ be nondecreasing sequences such that

$$g_i^*(n) \le \min\{n, g_i(n)\} \quad \textit{and} \quad d_i(n) \le n \le a_i(n) \quad \textit{for} \quad n \in P_+$$

$$g_i(n) \le d_i(n) \quad \textit{for} \quad n \in D$$

$$\textit{and} \quad a_i(n) \le g_i(n) \quad \textit{for} \quad n \in A.$$

Let $D_i(n) = D \cap [d_i(n), n]$ and $A_i(n) = A \cap [n, a_i(n)]$ for $n \in P_+$.

To obtain our main results we need the following two lemmas.

**Lemma 1.1** *[1] Let $x(n)$ be a sequence of real numbers. Let $\{x(n)\}$ and $\Delta^m x(n)$ be of constant sign with $\Delta^m x(n)$ not identically zero. Then there exists an integer $l \in \{0,1,2, \cdots, m\}$ and $n_0 > 0$ such that $m + l + z$ even and for $n \ge n_0$*

$$x(n)\Delta^j x(n) > 0, \quad j = 0,1,2, \cdots, l-1$$

$$\textit{and} \quad (-1)^{j+1} x(n)\Delta^j x(n) > 0, \quad j = 0,1,2, \cdots, m-1 \qquad (I_z).$$

**Lemma 1.2** *[14] Let $x(n)$ be a non-oscillatory solutions of $(E)$ satisfying the inequality (N) with $l \in \{1,2, \cdots, m-1\}$ and $m + l + z$ even. In addition, let*

$$\sum (n)^{(m-l)} \left|\Delta^m x(n)\right| = \infty. \tag{1}$$

The following inequalities hold for sufficiently large $n \ge n_1$ where $n_1 \ge n_0$

$$\left|\Delta^{l-1} x(n)\right| \ge \frac{(n-m+l)}{(m-l)!} \sum_{n_1}^{\infty} (s)^{(m-l-1)} \, |\Delta^m x(n)| \tag{2}$$

and

$$|x(s)| \ge \frac{(n-m+l)^{(l-1)}}{(l-1)!} |\Delta^{l-1} x(n)|. \tag{3}$$

## II. Main Results

We begin this section with the following theorem.

**Theorem 2.1** *Consider the difference inequality $(E_z)$ subject to the condition*

$$\limsup_{n\to\infty} \prod_{j=1}^{k} [\sum_{g_j^*(s)}^{n-1} p(s) \prod_{i=1}^{k} \left[g_i^*(s) - m + l\right]^{(m-1)\alpha_i}$$

$$+ \prod_{i=1}^{k} \left[g_i^*(n) - m + l\right]^{\alpha_i} \sum_{n}^{\infty} p(s) \prod_{i=1}^{k} \left[g_i^*(s) - m + l\right]^{(m-2)\alpha_i} ]^{\alpha_j} > (m-l)!. \tag{4}$$

Then [(i)]

1. for $m$ even, every solution of $(E_1)$ is oscillatory.
2. for $m$ odd, every solution of $(E_1)$ is either oscillatory, or $\lim_{n\to\infty} \Delta^j x(n)$, $(j = 0,1,2, \cdots, m-1)$ monotonically
3. for $m$ odd, every solution of $(E_2)$ is either oscillatory or $\lim_{n\to\infty} |\Delta^j x(n)| = \infty$, $(j = 0,1,2, \cdots, m-1)$ monotonically

4. for $m$ even, every solution of (E) is either oscillatory, or $\lim_{n\to\infty} \Delta^j x(n) = 0$ or $\lim_{n\to\infty} |\Delta^j x(n)| = \infty$, $(j = 0,1,2, \cdots, m-1)$ monotonically.

***Proof.*** Suppose that the inequality $(E_z)$ has a non-oscillatory solution $x(n) \neq 0$ for $n \geq n_0$. Therefore for sufficiently large $n \geq n_1$ by Lemma 1.1, there exists an integer $l \in \{0,1,2, \cdots, m\}$ with $m + l + z$ even, such that $x(n)$ satisfies the inequalities $(I_z)$.

**Case I:** In this case $m$ is even, then we have $z = 1$ and odd $l \in \{1,3, \cdots, (m-1)\}$. We observe that (4) and $(I_z)$ $(1 \leq l \leq m-1)$ imply that the condition (1) of Lemma 1.2 is satisfied. Therefore (2) and $(I_z)$ yield

$$|\Delta^{l-1} x(n)| \geq \frac{(n-m+l)}{(m-l)!} \sum_{n}^{\infty} (s)^{(m-l-1)} p(s) \prod_{i=1}^{k} |x(g_i(s))|^{\alpha_i}$$

$$\geq \frac{(n-m+l)}{(m-l)!} \sum_{n}^{\infty} (s)^{(m-l-1)} p(s) \prod_{i=1}^{k} |x(g_i^*(s))|^{\alpha_i}$$

which yields, by (3)

$$|\Delta^{l-1} x(n)| \geq \frac{(n-m+l)}{l!(m-l)!} \sum_{n}^{\infty} (s)^{(m-l-1)} p(s) \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(l-1)\alpha_i} |\Delta^{l-1} x(g_i^*(s))|^{\alpha_i} .$$

From the above inequality for $j \in \{1,2, \cdots, l\}$ and $n \geq n_1$ we get

$$l!(m-l)! \frac{|\Delta^{l-1} x(g_j^*(n))|}{g_j^*(n)} \geq \sum_{g_j^*(n)}^{n-1} (s)^{(m-l-1)} p(s) \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(l-1)\alpha_i} |\Delta^{l-1} x(g_i^*(s))|^{\alpha_i}$$

$$+ \sum_{n}^{\infty} (s)^{(m-l-1)} p(s) \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(l-1)\alpha_i} |\Delta^{l-1} x(g_i^*(s))|^{\alpha_i} \tag{5}$$

since $|\Delta^{l-1} x(n)| \times \frac{1}{n}$ is non-increasing for $1 \leq l \leq m-1$ and $n \geq n_1$ by (3) we obtain for $s \in [g_j^*(n), n]$ and $i \in \{1,2, \cdots, k\}$

$$|\Delta^{l-1} x(g_i^*(s))| \geq \frac{g_i^*(s)}{g_i^*(n)} |\Delta^{l-1} x(g_i^*(s))| . \tag{6}$$

Therefore from (5) and (6), in view of the increasing character of $|\Delta^{l-1} x(n)|$ we derive for $n \geq n_1$

$$l!(m-l)! \frac{|\Delta^{l-1} x(g_j^*(n))|}{g_j^*(n)} \geq \prod_{i=1}^{k} \left[ \frac{|\Delta^{l-1} x(g_i^*(n))|}{g_j^*(n)} \right]^{\alpha_i} \sum_{g_j^*(n)}^{n-1} (s)^{(m-l-1)} p(s)$$

$$\prod_{i=1}^{k} [g_i^*(s) - m + l]^{l\alpha_i} + |\Delta^{l-1} x(g_i^*(n))|^{\alpha_i} \sum_{n}^{\infty} (s)^{(m-l-1)} p(s) \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(l-1)\alpha_i} \tag{7}$$

for $l \in \{1,2, \cdots, (m-1)\}$, the following inequalities hold for $s \geq n_1$,

$$(s)^{(m-l-1)} \prod_{i=1}^{k} [g_i^*(s) - m + l]^{l\alpha_i} = (s)^{(m-1)} \prod_{i=1}^{k} \left[ \frac{g_i^*(s) - m + l}{s} \right]^{l\alpha_i}$$

$$\geq (s)^{(m-1)} \prod_{i=1}^{k} \left[ \frac{g_i^*(s) - m + l}{s} \right]^{(m-1)\alpha_i}$$

$$= \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(m-1)\alpha_i} \tag{8}$$

and

$$(s)^{(m-l-1)} \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(l-1)\alpha_i} \geq \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(m-2)\alpha_i} . \tag{9}$$

Using now (8) and (9) in (7), we get

$$(m-1)!\,\frac{|\Delta^{l-1}x(g_j^*(n))|}{g_j^*(n)-1} \geq \prod_{i=1}^{k}\left[\frac{|\Delta^{l-1}x(g_i^*(n))|}{g_j^*(n)}\right]^{\alpha_i} \times$$

$$\left\{\sum_{g_j^*(n)}^{n-1} p(s)\prod_{i=1}^{k}[g_i^*(s)-m+l]^{(m-1)\alpha_i} + \prod_{i=1}^{k}[g_i^*(n)-m+l]^{\alpha_i}\sum_{n}^{\infty} p(s)\prod_{i=1}^{k}[g_i^*(s)-m+l]^{(m-2)\alpha_i}\right\}.$$

Raising both sides of the above inequality to $\alpha_j$ and then multiplying the resulting inequalities, we have

$$(m-1)!\prod_{j=1}^{k}\left[\frac{|\Delta^{l-1}x(g_j^*(n))|}{g_j^*(n)}\right]^{\alpha_i} \geq \prod_{i=1}^{k}\left[\frac{|\Delta^{l-1}x(g_i^*(n))|}{g_j^*(n)-1}\right]^{\alpha_i} \times$$

$$\prod_{j=1}^{k}[\sum_{g_j^*(n)}^{n-1} p(s)\prod_{i=1}^{k}[g_i^*(s)-m+l]^{(m-1)\alpha_i} + \prod_{i=1}^{k}[g_i^*(n)-m+l]^{\alpha_i}\sum_{n}^{\infty} p(s)\prod_{i=1}^{k}[g_i^*(s)-m+l]^{(m-2)\alpha_i}]^{\alpha_j}$$

which contradicts (4).

**Case II:** Then $x(n)$ satisfies the inequalities $(I_z)$ of $l \in \{0,2,4, \cdots, m-1\}$. By arguments similar to those in the proof of Case (I), we prove the case $l \in \{2,4, \cdots, m-1\}$ is impossible. Therefore $x(n)$ satisfies $(I_z)$ for $l = 0$, that is,

$$(-1)^j x(n)\Delta^j x(n) > 0 \quad j = 0,1,2, \cdots, m-1, \quad n \geq n_1. \tag{10}$$

We shall prove that $\lim_{n\to\infty} x(n) = 0$. Suppose to the contrary that $\lim_{n\to\infty} x(n) = C > 0$. Then $|x(g_i(n))| \geq C$ $(i = 1,2, \cdots, k)$ for $n \geq n_2 \geq n_1$. From (10), it follows that

$$\sum_{n_2}^{\infty}(n)^{(m-1)}|\Delta^m x(n)| < \infty$$

which implies, by $(E_1)$

$$\infty > \sum_{n_2}^{\infty}(n)^{(m-1)}|\Delta^m x(n)| \geq \sum_{n_2}^{\infty}(n)^{(m-1)}p(n)\prod_{i=1}^{k}|x(g_i(n))|^{\alpha_i}$$

$$\geq C\sum_{n_2}^{\infty}(n)^{(m-1)}p(n).$$

But this gives a contradiction, since (4) implies that

$$\sum_{n_2}^{\infty} p(n)\prod_{i=1}^{k}[g_i^*(n)]^{(m-1)\alpha_i} = \infty. \tag{11}$$

**Case III and Case IV:** Then $x(n)$ satisfies $(I_z)$ for $l \in \{0,2,4, \cdots, m\}$. The case $l = 0$ holds only when $m$ is even. Then, by arguments analogous to those in the proof of II, we have $\lim_{n\to\infty}\Delta^j x(n) = 0$ $\{j = 0,1, \cdots, m-1\}$. Similarly as in the proof of Case I, we prove that the case $l \in \{2,4, \cdots, (m-2)\}$ is impossible. In the case $l = m$, we have

$$x(n)\Delta^m x(n) \geq 0 \quad \textit{and} \quad x(n)\Delta^j x(n) > 0, \quad (j = 0,1,2, \cdots, (m-1)) \tag{12}$$

for $n \geq n_2$. We shall prove that $\lim_{n\to\infty}|\Delta^j x(n)| = \infty$, $(j = 0,1, \cdots, (m-1))$. From (12) it follows that there exists a point $n_3 \geq n_2$ and a positive constant $\gamma$ such that

$$|x(g_i(n))| \geq \gamma(g_i(n))^{(m-1)}, \quad \textit{for} \quad n \geq n_3, \quad (i = 1,2, \cdots, k), \tag{13}$$

Summing now $(E_2)$ from $n_3$ to $n-1$ by (13) we obtain

$$|\Delta^{m-1}x(n)| \geq |\Delta^{m-1}x(n_3)| + \sum_{n_3}^{n-1} p(s)\prod_{i=1}^{k}|x(g_i(s))|^{\alpha_i}$$

$$\geq \gamma \sum_{n_3}^{n-1} p(s) \prod_{i=1}^{k} (g_i(s) - m + l)^{(m-1)\alpha_i}$$

$$\geq \gamma \sum_{n_3}^{n-1} p(s) \prod_{i=1}^{k} (g_i^*(s) - m + l)^{(m-1)\alpha_i}.$$

From the above inequality and (11), we get $\lim_{n\to\infty} |\Delta^j x(n)| = \infty$ , $(j = 0,1,2, \cdots, (m-1))$ . Thus the proof is complete. ■

**Corollary 2.1** *Consider the difference equation with general deviating argument*

$$\Delta^m x(n) + p(n) x(g(n)) = 0, \quad m \geq 2 \tag{14}$$

where $p(n)$ is the same as in $(E_2)$, $g : \mathbb{P}_+ \to \mathbb{P}_+$ is a sequence of integers such that $\lim_{n\to\infty} g(n) = \infty$ . Let the sequence $g^*(n) \leq \min\{n, g(n)\}$ be nondecreasing on $\mathbb{P}_+$ . If

$$\limsup_{n\to\infty} \left\{ \sum_{g^*(n)}^{n-1} (g^*(s) - m + l)^{(m-1)} p(s) + g^*(n - m + l) \sum_{n}^{\infty} (g^*(s) - m + l)^{(m-2)} p(s) \right\} > (n-1)! \tag{15}$$

then every solution of (14) is oscillatory, if $m$ is even and every solution of (14) is either oscillatory or $\lim_{n\to\infty} \Delta^j x(n) = 0,\ (j = 0,1,2, \cdots, (m-1))$ monotonically if $n$ is odd.

**Remark 2.1** *From corollary, in the case of ordinary linear difference equations* $(g(n) \leq n)$. *We obtain the results of [4]. In the case advanced difference equations* $(g(n) \geq n)$ *Corollary gives the result of [14].*

**Theorem 2.2** *If*

$$\limsup_{n\to\infty} (n - m + l) \prod_{j=1}^{k} \left[ \sum_{d_j(n)} (s - d_j(n))^{(m-l-1)} \prod_{i=1}^{k} (d_i(n) - g_i(s))^{(l-1)\alpha_i} p(s) \right]^{\alpha_j} > (m-1)!\,(l-1)! \tag{16}$$

for some $\in \{0,1, \cdots, (m-1)\}$ then every bounded solution of $(E_m)$ is oscillatory.

***Proof.*** Assume that $(E_m)$ has a bounded non-oscillatory solution $x(n) \neq 0$ for $n \geq n_0$. Then for sufficiently large $n \geq n_1 \geq n_0$, we have by $(E_m)$

$$(-1)^m x(n) \Delta^m x(n) \geq 0 \quad and \quad (-1)^j x(n) \Delta^j x(n) > 0, \quad (j = 0,1, \cdots, (m-1)). \tag{17}$$

From (17), as in [14] we obtain $u \geq n \geq n_1$

$$|\Delta^{l-1} x(n)| \geq \frac{(n - m + l)}{(m - l)!} \sum_{u=n}^{n} (u)^{(m-l-1)} |\Delta^m x(u)| \tag{18}$$

and

$$|x(s)| \geq \frac{(s-n)(m-1)}{(m-1)!} |\Delta^{m-1} x(n)|. \tag{19}$$

From (19) we have for $s \in D_j(n)$ $(j = 1,2, \cdots, k)$ and $n \geq n_1$

$$|x(g_i(s))| \geq \frac{(d_j(n) - g_i(s))^{(m-1)}}{(m-1)!} |\Delta^{m-1} x(d_i(n))|, \quad i = 1,2, \cdots, k. \tag{20}$$

Therefore from (17), $(E_m)$ and (20) we get for $n \geq n_1$ and $l \in (0,1, \cdots, (m-1))$ , $j \in (1,2, \cdots, k)$

$$|\Delta^{l-1} x(d_j(n))| \geq \frac{(n - m + l)}{(m - l)!} \sum_{d_j(n)}^{n-1} (s - d_j(n))^{(m-l-1)} |\Delta^m x(s)|$$

$$\geq \frac{(n - m + l)}{(m - l)!} \sum_{D_j(n)} (s - d_j(n))^{(m-l-1)} p(s) \prod_{i=1}^{k} |x(g_i(s)|^{\alpha_i}$$

$$\geq \prod_{i=1}^{k} |\Delta^{l-1} x(d_j(n))|^{\alpha_i} \frac{(n-m+l)}{(m-l)!\,(m-1)!} \sum_{D_j(n)} (s-d_j(n))^{(m-l-1)} p(s) \prod_{i=1}^{k} [d_j(n)-g_i(s)]^{(l-1)\alpha_i}.$$

Raising both sides of the above inequality ot $\alpha_j$ and then multiplying, we obtain

$$\prod_{j=1}^{k} |\Delta^{l-1} x(d_j(n))|^{\alpha_j} \geq \prod_{i=1}^{k} |\Delta^{l-1} x(d_j(n))|^{\alpha_i} \frac{(n-m+l)}{(m-l)!\,(m-1)!}$$

$$\prod_{j=1}^{k} [\sum_{D_j(n)} (s-d_j(n))^{(m-l-1)} p(s) \prod_{i=1}^{k} [d_j(n)-g_i(s)]^{(l-1)\alpha_i}]^{\alpha_j}$$

which contradicts (16). Thus the proof is complete. ■

**Corollary 2.2** *Suppose that in the equation* $(E_m)$, $g_i(n) \leq n$ $(i = 1,2, \cdots, k)$ *on* $\mathrm{P}_+$. *Then every bounded solution of* $(E_m)$ *is oscillatory, if*

$$\limsup_{n\to\infty} (n-m+l) \prod_{i=1}^{k} \left[ \sum_{d_i(n)}^{n-1} (s-d_i(n))^{(m-1)} p(s) \right]^{\alpha_i} > (m-1)!. \quad (21)$$

**Remark 2.2** *A condition similar to that in Corollary 2.2 for delay difference equation can be found in [4]. From the result it follows that every bounded solution of* $(E_m)$ *is oscillatory if*

$$\limsup_{n\to\infty} \{ \sum_{s=n-k}^{n} (s-n+k+m-1)^{(m-1)} p(s) + \left[ (n-k+m) + \frac{1}{(m-1)!} \sum_{n_1}^{n-k-1} (s-k)^{(m)} p(s) \right]$$

$$\times \sum_{s=n+1}^{\infty} (s-n+k+m-2)^{(m-2)} p(s) \} > (m-1)!. \quad (22)$$

We note that the condition (21) of Corollary 2.2 is better than (22).

**Theorem 2.3** *Let* $m \geq 3$ *be odd. Consider the difference inequality* $(E_1)$ *subject to the conditions (4) and (16). Then every solution of* $(E_1)$ *is oscillatory.*

***Proof.*** The above theorem follows from Theorems 2.1 and 2.2. ■

**Theorem 2.4** *Let* $m \geq 3$ *be odd. Consider the difference inequality* $(E_2)$ *subject to the conditions*

$$\limsup_{n\to\infty} (n-m+l) \prod_{j=1}^{k} [\sum_{g_i^*(n)}^{n-1} sp(s) \prod_{i=1}^{k} [g_i^*(s)-m+l]^{(m-2)\alpha_i}$$

$$+ \prod_{i=1}^{k} [g_i^*(n)-m+l]^{\alpha_i} \sum_{n}^{\infty} sp(s) \prod_{i=1}^{k} [g_i^*(s)-m+l]^{(m-3)\alpha_i}] > (m-1)! \quad (23)$$

and

$$\limsup_{n\to\infty} \prod_{j=1}^{k} [\sum_{A_j(n)} (a_j(s))^{(m-l-1)} \prod_{i=1}^{k} [g_i^*(s)-a_i(n)]^{(l-1)\alpha_i} p(s)]^{\alpha_j} > (m-l)!\,(l-1)! \quad (24)$$

for some $l \in \{0,1, \cdots, (m-1)\}$. Then every solution of $(E_2)$ is oscillatory.

***Proof.*** Suppose that the inequality $(E_2)$ has a non-oscillatory solution $x(n) \neq 0$ for $n \geq n_0$. Then Lemma 1.1 implies that either

$$x(n)\Delta^m x(n) \geq 0, \quad x(n)\Delta^j x(n) > 0, \quad (j = 0,1, \cdots, m-1) \quad (25)$$

or there exists an odd $l \in \{1,3, \cdots, m-2\}$ such that

$$x(n)\Delta^j x(n) > 0, \quad j = 0,1, \cdots, l-1$$

$$(-1)^{j+l} x(n)\Delta^j x(n) > 0, \quad j = l, l+1, \cdots, m-1 \tag{26}$$

for $n \geq n_1 \geq n_0$ . Let (25) hold. Then for $u \geq n \geq n_1$ and $l \in \{0,1, \cdots, (m-1)\}$

$$|\Delta^{l-i} x(u)| \geq \frac{u-m+l}{(m-l)!} \sum_{s=n}^{u} (s)^{(m-l-1)} |\Delta^m x(s)| \tag{27}$$

and

$$|x(u)| \geq \frac{(u-n)^{(l-1)}}{(l-1)!} |\Delta^{l-1} x(n)|. \tag{28}$$

Then from (28) for $s \in A_j(n)$ $(j = 1,2, \cdots, k)$ and $n \geq n_1$ , we obtain

$$|x(g_i(s))| \geq \frac{[g_i(s) - a_i(n)]^{(l-1)}}{(l-1)!} |\Delta^{l-1} x(a_i(n))|, \quad (i = 1,2, \cdots, k). \tag{29}$$

Then from (27), $(E_2)$ and (29), we derive for $j \in \{1,2, \cdots, k\}$ ,
$l \in \{1,2, \cdots, (m-1)\}$ and $n \geq n_1$

$$|\Delta^{l-1} x(a_j(n))| \geq \frac{(n-m+l)}{(m-l)!} \sum_{n}^{a_j(n)-1} (a_j(s))^{(m-l-1)} |\Delta^m x(s)|$$

$$\geq \frac{(n-m+l)}{(m-l)!} \sum_{A_j(n)} (a_j(s))^{(m-l-1)} p(s) \prod_{i=1}^{k} |x(g_i(s))|^{\alpha_i}$$

$$\geq \frac{(n-m+l)}{(l-1)!\,(m-l)!} \prod_{i=1}^{k} |\Delta^{l-1} x(a_j(n))|^{\alpha_i} \sum_{A_j(n)} (a_j(s))^{(m-l-1)} p(s)$$

$$\times \prod_{i=1}^{k} [g_i(s) - a_i(n)]^{(l-1)\alpha_i}.$$

Raising both sides of the above inequality ot $\alpha_j$ and then multiplying the resulting inequalities, we get

$$\prod_{j=1}^{k} |\Delta^{l-1} x(a_j(n))|^{\alpha_j} \geq \frac{(n-m+l)}{(l-1)!\,(m-l)!} \prod_{i=1}^{k} |\Delta^{l-1} x(a_i(n))|^{\alpha_i}$$

$$\prod_{j=1}^{k} \left[ \sum_{A_j(s)} (a_j(s))^{(m-l-1)} p(s) \prod_{i=1}^{k} [g_i(s) - a_i(n)]^{(l-1)\alpha_i} \right]^{\alpha_j}$$

which contradicts (24). Thus, the case (25) is impossible.

Suppose now that (26) holds. Then in view of (23) and (26), the assumptions of Lemma 1.2 are satisfied. Therefore, by arguments similar to those in the proof of Theorem 2.1 we obtain the inequality (7). Since $l \in \{1,3, \cdots, (m-2)\}$ , we get for $s \geq n_1$

$$(s)^{(m-l-1)} \prod_{i=1}^{k} [g_i^*(s) - m + l]^{l\alpha_i} = (s)^{(m-1)} \prod_{i=1}^{k} \left[ \frac{g_i^*(s) - m + l}{s} \right]^{l\alpha_i}$$

$$\geq (s)^{(m-1)} \prod_{i=1}^{k} \left[ \frac{g_i^*(s) - m + l}{s} \right]^{(m-2)\alpha_i}$$

$$= s \prod_{i=1}^{k} \left[ g_i^*(s) - m + l \right]^{(m-2)\alpha_i} \tag{30}$$

and

$$(s)^{(m-l-1)} \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(l-1)\alpha_i} = (s)^{(m-2)} \prod_{i=1}^{k} \left[ \frac{g_i^*(s) - m + l}{s} \right]^{(l-1)\alpha_i}$$

$$\geq (s)^{(m-2)} \prod_{i=1}^{k} \left[ \frac{g_i^*(s) - m + l}{s} \right]^{(m-3)\alpha_i}$$

$$= s \prod_{i=1}^{k} \left[ g_i^*(s) - m + l \right]^{(m-3)\alpha_i}. \qquad (31)$$

Using (30) and (31) in (7), we obtain

$$(n - m + l)(m - 1)! \frac{|\Delta^{l-1} x(g_j^*(n))|}{g_j^*(n)} \geq \prod_{i=1}^{k} \left[ \frac{|\Delta^{l-1} x(g_i^*(n))|}{g_i^*(n)} \right]^{\alpha_i}$$

$$\left\{ \sum_{g_j^*(n)}^{n-1} sp(s) \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(m-2)\alpha_i} + \prod_{i=1}^{k} [g_i^*(n) - m + l]^{\alpha_i} \sum_{n}^{\infty} sp(s) \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(m-3)\alpha_i} \right\}.$$

Proceeding as in the corresponding part of the proof of Theorem 2.1, we get a contradiction with the assumption (23). Thus the inequalities (23) cannot hold. This completes the proof. ■

**Theorem 2.5** *Let* $m$ *be even. Consider the difference inequality* $(E_2)$ *subject to the conditions (16) and (24). In addition, let for* $m \geq 4$

$$\limsup_{n \to \infty} (n - m + l) \prod_{j=1}^{k} [\sum_{g_i^*(n)}^{n-1} sp(s) \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(m-2)\alpha_i}$$

$$+ \prod_{i=1}^{k} [g_i^*(n) - m + l]^{\alpha_i} \sum_{n}^{\infty} sp(s) \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(m-3)\alpha_i}]^{\alpha_j} > 2(m - 2)!. \qquad (32)$$

Then every solution of $(E_2)$ is oscillatory.

***Proof.*** Let $x(n)$ be a non-oscillatory solution of $(E_2)$ for $n \geq n_0$. Thus by Lemma 1.1, $x(n)$ satisfies the inequalities $(I_z)$ with $l \in \{0, 2, \cdots, (m-2), m\}$. The case $l = 0$ and $l = m$ are impossible, by the assumptions (16), (24) and Theorem 2.2 and Theorem 2.4 respectively.
Suppose now that $l \in \{2, \cdots, (m-2)\}$ which is possible only if $n \geq 4$. Therefore by arguments similar to those in the corresponding part of the proof of Theorems 2.1 and 2.4, we obtain the inequalities (7), (30) and (31). Combining (30) and (31) with (7) and using the fact that $2 \leq l \leq m-2$, we have

$$2(m-2)!\ (n - m + l) \frac{|\Delta^{l-1} x(g_j^*(n))|}{g_j^*(n)} \geq \prod_{i=1}^{k} \left[ \frac{|\Delta^{l-1} x(g_i^*(n))|}{g_i^*(n)} \right]^{\alpha_i}$$

$$\sum_{g_j^*(n)}^{n-1} sp(s) \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(m-2)\alpha_i} + \prod_{i=1}^{k} |\Delta^{l-1} x(g_i^*(n))|^{\alpha_i} \sum_{n}^{\infty} sp(s) \prod_{i=1}^{k} [g_i^*(s) - m + l]^{(m-3)\alpha_i}.$$

From this inequality, similarly as in the proof of Theorem 2.4, we obtain a contradiction with the assumption (32). Thus $l \notin (2, 4, \cdots, (m-2))$ and the proof is complete. ■

## III. Final Remarks

The For simplicity, we consider the linear difference equation with a deviating argument

$$\Delta^m x(n) = p(n) x(g(n)) \qquad (L)$$

where $m$ is even, $p(n)$ is a positive real sequence and $g$ is nondecreasing and $\lim_{n \to \infty} g(n) = \infty$.

Let $g_*(n) = \min(n, g(n))$ $g^*(n) = \max(n, g(n))$ $D = \{x \in P_+ : g(n) < n\}$ and $A = \{n \in P_+ : g(n) > n\}$.
It is known that in the case of ordinary difference equation, that is $g(n) = n$, the equation (L) always has non-oscillatory solutions satisfying the inequalities $(I_0)$ and $(I_m)$. The situation is different when $g(n) \neq n$. For

example, in view of Theorem 2.1 and Theorem 2.2 every solution of (L) is either oscillatory or $\lim_{n\to\infty} |\Delta^l x(n)| = \infty \quad (l = 0,1, \cdots, (m-1))$ monotonically if the following conditions hold

$$\limsup_{n\to\infty} (n-m+l)\{\sum_{g^*(n)-m+l}^{n-1} [g_*(s)-m+l]^{(m-1)} p(s) + [g_*(n)-m+l]$$

$$\sum_{n}^{\infty} [g_*(s)-m+l]^{(m-2)} p(s)\} > (m-1)! \tag{33}$$

and for some $(l \in 0,1, \cdots, (m-1))$ ,

$$\limsup_{n\to\infty} (n-m+l) \sum_{D\bigcap[g_*(n),n]} [s-g_*(n)]^{(m-l-1)} [g_*(n)-g(s)]^{(l-1)} p(s) > (l-1)!\,(m-l)!. \tag{34}$$

On the basis of Theorems 2.1 and 2.4, we can prove that every solution $x(n)$ of (L) is either oscillatory or $\lim_{n\to\infty} \Delta^j x(n) = 0 \quad (j = 0,1, \cdots, (m-1))$ monotonically if (33) holds and

$$\limsup_{n\to\infty} (n-m+l) \sum_{A\bigcap[n,g^*(n)]} [g^*(n)-s]^{(m-l-1)} [g(s)-g^*(n)]^{(l-1)} p(s) > (l-1)!\,(m-l)!. \tag{35}$$

From Theorem 2.5, if follows that every solution of (L) is oscillatory fi (34) and (35) hold. In addition, when $m \ge 4$ the following inequality is satisfied

$$\limsup_{n\to\infty} (n-m+l) \sum_{g^*(n)}^{n-1} s[g_*(s)-m+l]^{(m-2)} p(s) + [g_*(n)-m+l] \sum_{n}^{\infty} s[g_*(s)-m+l]^{(m-3)} p(s) > 2(m-2)!.$$

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