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Exponential Stabilizability and Robustness of Control Systems

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Abstract: *In this paper, some results on the exponential stabilizability and robustness analysis for non-linear control system were derived. Also derived was an important absolute estimate for the robustness index of the non linear control systems.*

Key Words: *Exponential Stabilization, Robustness, Linearization, Perturbation.*

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I. Introduction

Consider the control system of the form

$$\dot{x} = f(x,u),\ x(0) = x_0 \tag{1}$$

with the set $U \subset E^m$, the m-dimensional Echidum space, is a control parameters; where E^m is containing the origin O and the function $f: E^n \times U \to E^n$ *is such that* $f(0,0) = 0$. We know that by feedback function, we mean an arbitrary continuous function $u(.) : E^n \to U,\ u(0) = 0$ such that the closed-loop system.

$$\dot{x} = g(x),\ x(0) = x_0 \tag{2}$$

where

$$g(x) = f(x\ ,u(x)),\ x \epsilon E^n \tag{3}$$

has a unique solution $x(t,x_0), t \geq 0$, which depends continuously on the initial data x_0.

We can talk of a feedback u(.) which is stabilizing if 0 is an isolated stable equilibrium for (2). Now, a stabilizability problem is formulated as follows:

1) Can we find checkable conditions on the right hand side of (1) which guarantees the existence of a stabilizing feedback?
2) Can we propose a method of constructing a stabilizing feedback?
3) Can we indicate how to select a feedback from the set of all stabilizing feedbacks, which stabilizes (1) in an optimal way?

Consider the linear control system.

$$\dot{x} = Ax + Bu,\ \ x(0) = x_0\ \epsilon\ E^n \tag{4}$$

We say that (4) is stabilizable or that the pair (A, B) is stabilizable if there exists a matrix K such that the matrix A + KB is stable. So, if the pair (A, B) is stabilizable and the control u(.) is given in the feedback form.

$$i.e.\quad u(t) = K\ x(t),\ t \geq 0 \tag{5}$$

then all solutions of the equation

$$\dot{x}(t) = Ax(t) + BKx(t) = (A + BK)\ x(t)$$
$$x(0) = x_0,\ t \geq 0 \tag{6}$$

tends to zero as $t \to \infty$ [4]

We say that system (4) is exponential stabilizable if and only if for arbitrary number $\alpha > 0$ there exist a matrix K and a constant $M > 0$ such that for an arbitrary solution $x(t,x_0),\ t \geq 0$ of (6), we have

$$|x(t,x_0)|\ \leq Me^{-\alpha t}\,|x_0|,\ t \geq 0 \tag{7}$$

We note that exponential stabilizability of (1) is equivalent to exponential stabilizability of its linearization. This implies that the solution to the points 1and 2 of the stabilizability questions exist. Then we face the point 3 of the questions, so proving the main result of our problem.

As is well known, controllability of a linear system implies it stabilizability. Thus the rank condition implies the existence of a stabilizing feedback. Note that generalization of this result to non-linear systems would have been great values, as for controllability, we could have expected as in many important cases, checkable algebraic conditions for stabilizability. However, as was pointed out by Sussmann [1] and Brokett [2], contrary to popular belief, controllability of non-linear system does not, in any way, imply their stabilizability.

II. Exponential Stabilizability and Robustness.

Here we require that the origin 0 is an interior point of U and that the function f and admissible feedback u(.) are of class C^1-function. By linearization of (1), we mean the following linear system.

$$\dot{x} = Ax + Bu \quad (8)$$

where $A = f_x(0,0)$ and $B = f_u(0,0)$. According to [6], system is said to be exponential stabilizable if there exist an admissible feedback u(t) and constants $\alpha > 0,\ M > 0$ such that for an arbitrary solution $x(t,0),\ t \geq 0$ of (8) we have

$$|x(t,x_0)| \leq Me^{-\alpha t}|x_0| \quad (9)$$

We note that system (1) is exponential stabilizable if and only if the linearization (8) is exponentially stabilizable. Moreover, if system (8) is stabilizable, then a stabilizing feedback for (1) can be the linear part of it. A proof that stabilizability of linearization implies stabilizability of the initial system (1) can be seen in Lee and Markus [3].

The following proposition shall be of immense benefit for us.

Proposition 1:

Assume that the condition (9) holds for some positive numbers α and β with x_0 sufficiently close to the origin 0, and for all $t \geq 0$, then for arbitrary positive constants N > M and $\beta < \alpha$,

$$|e^{At}| < Ne^{-\beta t}\ for\ all\ t \geq 0 \quad (10)$$

Now, let us go back to the controlled system (1) and if u(.) is a feedback stabilizing (1) exponentially and $g(x) = f(x, u(x)),\ x \in E^n, then$

$$g(x) = \big(f_x(0,0), f_u(0,0)\big)\, U_x(0)x \ + \ h(x),\ x \in E^n\ with\ \frac{h(x)}{|x|} \to 0\ as\ x \to 0.$$

So, the pair $\big(f_x(0,0), f_u(0,0)\big)$ is stabilizable. Also, if $K = U_x(0)$ and $u(.)$ is a C^1-feedback such that $U_x(0) = K$, then u(.) also exponentially stabilizes (1). This is especially true for linear feedback where $u(x) = Kx$, $x \in E^n$. Then from the note on exponentially stabilizability and linearization of non-linear systems, we have the following important implication.

Implication

If the linearization (8) of system (1) is not stabilizable, then feedback stabilizing (1) if they exist at all, has a slow rate of convergence to the trajectories of system (2) to zero 0. So, exponential stabilizability is a very strong property. However, even for exponentially stabilizing system, there are absolute limits as to how well they will behave. This leads us to the limitation connected with the so called robustness index.

Now suppose that the exponential stabilizability of system (2) is such that condition (7) holds, we consider a perturbed system.

$$\dot{x} = g(x) + r(x) \quad (11)$$

where r(.) is a C^1-function such that r(0) = 0. We now ask a natural question. How large could the perturbation r(x) be so that the exponential stabilizability the initial system (2) will be maintained? This leads the following proposition.

Proposition 2. [5]

If $\lim_{x \to 0} \left|\frac{r(x)}{|x|}\right| < \frac{\alpha}{M}$, then the perturbed system (11) is exponentially stabilizable.

III. Robustness Index:

The maximum of the ratio $\frac{\alpha}{M}$ so that the preposition 2 holds is called the robustness index of (2).

For system (11) we assume that $g_x(0) = A$ and $r_x(0) = C$. Then the linearization of (11) will be of the form

$$\dot{x} = [A + C]\, x \quad (12)$$

We note that $\lim_{x \to 0} \frac{|Cx|}{|x|} \leq \lim_{x \to 0} \frac{|r(x)|}{|x|} < \frac{\alpha}{M}$.

Then $|c| < \frac{\beta}{N}\ for\ \beta < \alpha.$ and N > M,

These are sufficiently close to α and M respectively such that

$$|e^{At}| < Ne^{-\beta t}\ for\ all\ t \geq 0\ .$$

As is obviously true, the system (12) is exponentially stabilizable and hence system (11). So we have the following preposition.

Preposition 3. [5]

Assume $R(A) \leq R(B),\ then$

$\frac{\alpha}{M} \leq \frac{|A|}{d}\ , with\ d > 0$. If in addition $R(A) = E^n\ , then\ {}^{\alpha}/_{M} \leq |A|$.

If $R(A) \subseteq R(B)$ and the pair (A, B) is controllable, then we can stabilize system (1) with the decay rate α as large as we wish, but then M also increases to infinity which is not proper.We are now ready to state our main theorem thus:

Theorem.

For arbitrary feedback u(.) exponentially stabilizing (1) such that (7) holds, we have

$$\alpha/_M \leq \frac{A}{d} \tag{13}$$

where $d = \sup_{x\epsilon\, R(A)} \rho_{|x|=1}\left(x, R(B)\right)$

where R(A) and R(B) denotes the ranges of transformation of A and B, and by convection $\frac{x}{0} = \frac{a}{0} = +\infty \; for \; a > 0.$

Proof:

From proposition 1 and comments following it, we are bent towards limiting our considerations to linear systems and stabilizing linear feedbacks $u(x) = Kx,\ x \,\epsilon\, E^n$. Thus we assume

$\dot{x} = Ax + Bu,$ and let $\bar{x} \,\epsilon\, E^n$ be a point such that $A\bar{x} + Bu \neq 0$ for all $u \,\epsilon\, E^m$. Let us define control $\bar{u}(t) = K(x(t) - \bar{x}),\ t \geq 0$, where $x(t)$ is the response to $\bar{u}(t)$ from x such that

$$\dot{x} = Ax + B\bar{u},\ x(0) = \bar{x},\ t \geq 0.$$

Then we have for $y(t) = x(t) - \bar{x},\ t \geq 0$

$$\dot{y} = (A + BK)\, y + A\bar{x},\ \ y(0) = 0.$$

then

$$y(t) = \int_0^t e^{(A+BK)\,(t-s)}\ A\bar{x}\, ds,\ t \geq 0$$

The inequality

$$\left|e^{(A+BK)\,t}\right| \leq Me^{\alpha t}, t \geq 0 \text{ implies}$$

that

$$|y(t)| \leq \frac{M}{\alpha}\ |A\bar{x}| \text{ for all } t \geq 0.$$

On the other hand, let v be a vector of the form $A\bar{x} + B\bar{u}$ having the minimal norm, then $|v| > 0, and\ for\ all\ u \,\epsilon\, E^m, \langle A\bar{x} + Bu - v, v\rangle = 0$

Consequently, for arbitrary $x \,\epsilon\, E^n\ and\ u \,\epsilon\, E^m$,

$$\langle Ax + Bu,\ v\rangle + \langle A\bar{x} + Bu + A(x - \bar{x}),\ v\rangle$$
$$= |v^2| + \langle A\,(x - \bar{x},\ v\rangle \geq |v|^2 - |A|\ |x - \bar{x}|\,|v|.$$

Let us take δ to an arbitrary number such that $|v|^2 > \delta > 0$. Then we can see that if

$$|x - \bar{x}| < \frac{1}{A}\left(|v| - \frac{\delta}{|v|}\right) \tag{14}$$

For arbitrary $\delta,\ |v|^2 > \delta > 0$, then

$$\frac{1}{A}\left(|v| - \frac{\delta}{|v|}\right) \leq \frac{M}{\alpha}\ |A\bar{x}|. \tag{15}$$

Eventually, taking in (15) infimum of the right-hand side with respect to all $A\bar{x},\ i.e.\ |A\bar{x}| = 1$, we arrive at the required inequality taking $|v| = d$.

IV. Conclusion

We conclude that exponential stabilizability is a very strong property of a control system. A system which can be stabilized exponentially has an absolute limit as how well it will behave. This absolute limit, called robustness index will not be exceeded for proper behaviour.

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