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Optimal Decision Making Method Using Interval Valued Intuitionistic Fuzzy Divergence Measure Based On The Weights Of Alternatives

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Abstract: It has been observed that the divergence measures for the interval valued intuitionistic fuzzy sets play a significant role towards applications in a variety of disciplines. The present communication is a step in the direction of constructing such a measure of divergence along with the study of its detailed properties for its validity. The applications of this newly developed fuzzy divergence measure have been provided to the optimal decision making based on the weights of alternatives. Numerical verification has been illustrated to demonstrate the proposed method for solving optimal decision making problem under fuzzy environment.

Keywords: *Fuzzy sets, Fuzzy divergence measure, Intuitionistic Fuzzy cross-entropy, Interval valued intuitionistic fuzzy cross-entropy, Optimal decision making.*

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I. Introduction

Today, one of the most complex administrative processes in management is the decision making and is the process to find the best substitute among a set of feasible alternatives. As pointed out by Wei, Zhao & Lin (2013), it may involve some conflicting and incommensurable attributes. However, due to the complexity present in the decision system and the lack of knowledge a decision maker may provide his/her preferences over alternatives with incomplete information and imprecise preferences. Fuzzy set theory introduced by Zadeh (1965) is one of the most suitable procedures to tackle with such situations. After the foreword of fuzzy sets (FS) by Zadeh (1965), many approaches and theories treating imprecise information were proposed by Zadeh (2005), (2008). Out of these fuzzy sets, intuitionistic fuzzy sets (IFSs) were proposed by Atanassov (1986) and these sets were extended by Atanassov and Gargov (1989) to the interval-valued intuitionistic fuzzy sets (IVIFSs). These sets have been characterized by a membership function and a non-membership function whose values are intervals rather than exact numbers.

This is to add that an IVFS is recommended to specify an interval-valued degree of membership to each element of the universe, and an intuitionistic fuzzy set allocates both a membership μ and a non-membership ν to each element of the universe such that $0 \le \mu + \nu \le 1$. Furthermore, the concept of vague set introduced by Gau (1993) is another extension of ordinary fuzzy set, and Bustince and Burillo (1996) has proved its equivalence to IFS. It may be worth mentioning here that both types of sets have been accomplished in various areas of research including those approximate reasoning, pattern recognition and decision-making. As remarked by Atanassov and Gargov (1989) an IVIFS can also be described by a membership interval, a non-membership interval and a hesitance interval. Thus, the interval-valued intuitionistic fuzzy set has the virtue of complementing fuzzy set and IFS, which is more flexible and practical than FS and IFS in coping with fuzziness and uncertainty.

In the literature of information theory, the measure of divergence, first introduced by Kullback and Leibler (1951) plays an important role because of its applications to a variety of disciplines and it is a measure of the extent to which the assumed probability distribution deviates from the true one. After the introduction of fuzzy sets, a large number of researchers studied the divergence measures for fuzzy distributions in different ways and provided their applications in different disciplines of mathematical sciences. Bhandari and Pal (1993) proposed a measure of fuzzy divergence between two fuzzy sets corresponding to Kullback-Leibler's (1951) measure of divergence. Fan and Xie (1999) introduced the fuzzy divergence measure based on exponential operation and studied its relation with fuzzy divergence measure introduced by Bhandari and Pal (1993). Ghosh et al. (2010) provided its applications in the area of automated leukocyte recognition whereas Montes et al. (2002) studied the special classes of divergence measures and used the link between fuzzy and probability uncertainty. Parkash (2000) introduced a symmetric divergence measure for fuzzy distributions whereas Parkash and Sharma (2005), and Parkash et al. (2006) proposed some new fuzzy divergence measures and studied their detailed properties.

The present paper is organized as follows:

In section 2, some basic definitions related to the literature of fuzzy set theory have been provided. In section 3, we propose a new fuzzy divergence measure of a fuzzy set with respect to another fuzzy set and study some more elegant properties of the proposed fuzzy divergence measure in the form of theorems. In section 4, we propose fuzzy cross-entropy of interval valued intuitionistic fuzzy sets (IVIFSs) and algorithm to solve optimal decision making method based on the weights of alternatives is discussed. Finally, in section 5, numerical verification has been presented to illustrate the procedure of proposed algorithm to solve optimal decision making method based on the weights of alternatives.

II. Preliminiries

In this section, we provide some basic concepts and notions related to fuzzy set theory necessary for the analysis under consideration.

2.1. Fuzzy Set Theory: This is to be emphasised that fuzziness is a feature of uncertainty and the boundaries of the fuzzy set under consideration are not sharply defined.

Definition 2.1. *Fuzzy Set:* A fuzzy set A defined in a finite universe of discourse $X=\{x_1,x_2,...,x_n\}$ is mathematically expressed as

$$A=\left\{\left\langle x,\ \mu_A(x)\right\rangle \middle|\ x\in X\ \right\} \tag{2.1}$$

where $\mu_A(x): X\to[0,\,1]$ is measure of belongingness of degree of membership of an element $x\in X$ in A.

Definition 2.2. *Set Operations on Fuzzy Sets:* Let $FS(X)$ denote the family of all fuzzy sets (FSs) in the universe X, and let $A,B\in FS(X)$ be two FSs, given by

$A=\left\{\left\langle x,\ \mu_A(x)\right\rangle \middle|\ x\in X\ \right\}$ and $B=\left\{\left\langle x,\ \mu_B(x)\right\rangle \middle|\ x\in X\ \right\}$, then following set operations hold on FSs:

(i) $A^C=\left\{\left\langle x,1-\mu_A(x)\right\rangle \middle|\ x\in X\ \right\}$;

(ii) $A\cap B=\left\{\left\langle x,\ \mu_A(x)\wedge\mu_B(x)\right\rangle \middle| x\in X\ \right\}$;

(iii) $A\cup B=\left\{\left\langle x,\ \mu_A(x)\vee\mu_B(x)\right\rangle \middle| x\in X\ \right\}$;

where $\vee,\ \wedge$ stand for maximum and minimum operators, respectively.

Afterwards, Atanassov (1986, 1999) introduced the following generalization of fuzzy sets, called intuitionistic fuzzy sets.

Definition 2.3. *Intuitionistic Fuzzy Sets*: An intuitionistic fuzzy set A in a finite universe of discourse $X=\{x_1,x_2,...,x_n\}$ is defined as

$$A=\left\{\ \left\langle x,\ \mu_A(x),\nu_A(x)\right\rangle \middle|\ x\in X\ \right\}$$

where $\mu_A(x): X\to[0,\,1]$ and $\nu_A(x): X\to[0,\,1]$ with the condition that $0\le\mu_A(x)+\nu_A(x)\le 1$. For each $x\in X$, the numbers $\mu_A(x)$ and $\nu_A(x)$ denote the degree of membership and degree of non-membership of x to A respectively. Further, $\pi_A(x)=1-\mu_A(x)-\nu_A(x)$ is called the degree of hesitance or intuitionistic index $x\in X$ to A.

Moreover, Xu (2007) introduced two weighted aggregation operators related to IVIFSs through the following definitions:

Definition 2.4. Let $A_j\ (j=1,2,...,n)\in\text{IVIFS}(X)$. The weighted arithmetic average operator is defined as

$$F_w(A_1,A_2,...,A_n)=\left(\left[1-\prod_{j=1}^{n}\left(1-\mu_{A_jL}(x)\right)^{w_j},1-\prod_{j=1}^{n}\left(1-\mu_{A_jU}(x)\right)^{w_j}\right],\left[\prod_{j=1}^{n}\nu_{A_jL}^{w_j}(x),\prod_{j=1}^{n}\nu_{A_jU}^{w_j}(x)\right]\right) \tag{2.2}$$

where w_j is the weight of $A_j\left(j=1,2,...,n\right)$, $w_j\in\left[0,1\right]$ and $\sum_{j=1}^{n}w_j=1$. Especially, assume $w_j=\frac{1}{n}\ \left(j=1,2,...,n\right)$, then F_w is called an arithmetic operator for IVIFSs.

Definition 2.5. Let $A_j\left(j=1,2,...,n\right)\in\text{IVIFS}\left(X\right)$. The weighted geometric average operator is defined as

$$G_w\left(A_1,A_2,...,A_n\right)=\left(\left[\prod_{j=1}^{n}\mu_{A_jL}^{w_j}\left(x\right),\prod_{j=1}^{n}\mu_{A_jU}^{w_j}\left(x\right)\right],\left[1-\prod_{j=1}^{n}\left(1-\nu_{A_jL}\left(x\right)\right)^{w_j},1-\prod_{j=1}^{n}\left(1-\nu_{A_jU}\left(x\right)\right)^{w_j}\right]\right) \tag{2.3}$$

where w_j is the weight of $A_j\left(j=1,2,...,n\right)$, $w_j\in\left[0,1\right]$ and $\sum_{j=1}^{n}w_j=1$. Especially, assume $w_j=\frac{1}{n}\ \left(j=1,2,...,n\right)$, then G_w is called an geometric average operator for IVIFSs.

The aggregation results F_w and G_w are still IVIFSs. Obviously, there are different emphasis points between (2.2) and (2.3). The weighted arithmetic average operator emphasizes the group's influence, so it is not very sensitive to $A_j\left(j=1,2,...,n\right)\in\text{IVIFS}\left(X\right)$, whereas the weighted geometric average operator emphasizes the individual influence, so it is more sensitive to $A_j\left(j=1,2,...,n\right)\in\text{IVIFS}\left(X\right)$.

Couso et al. (2000) defined that if X is a universe of discourse and $F\left(X\right)$ is the set of all fuzzy subsets, a mapping $D:F\left(X\right)\times F\left(X\right)\rightarrow R$ is a divergence measure if and only if for each $A,B,C\in F\left(X\right)$, the following axioms hold:

(a) $D\left(A,B\right)=D\left(B,A\right)$

(b) $D\left(A,A\right)=0$

(c) $\max\left\{D\left(A\cup C,\ B\cup C\right),\ D\left(A\cap C,\ B\cap C\right)\right\}\le D\left(A,B\right)$.

The non-negativity of the divergence is not required in the previous axioms but it is trivial to deduce it from axioms (b) and (c).

III. A New Fuzzy Divergence Measure

Let A and B be two fuzzy sets defined in a finite universe of discourse $X=\{x_1,x_2,...,x_n\}$ having the membership values $\mu_A\left(x_i\right),\ \ i=1,2,3,...,n$ and $\mu_B\left(x_i\right),\ \ i=1,2,3,...,n$ respectively. Then, we propose a new fuzzy divergence measure of fuzzy set B with respect to fuzzy set A, as follows:

$$K\left(A,B\right)=-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_A\left(x_i\right)\,\mu_B\left(x_i\right)}+\sqrt{\left(1-\mu_A\left(x_i\right)\right)\left(1-\mu_B\left(x_i\right)\right)}\right]}{2}\right) \tag{3.1}$$

Theorem 3.1: $K\left(A;\ B\right)$ is a valid fuzzy divergence measure.

Proof: From (3.1), it is understood that

(i) $K\left(A,B\right)=K\left(B,A\right)$;

(ii) $K\left(A,B\right)=0$ if and only if $\mu_A\left(x_i\right)=\mu_B\left(x_i\right)$

(iii) Now, we check the validity of axiom (c) of definition of Couso et al. (2000).

We divide X into the following six subsets:

$X_1=\{x/x\in X,\mu_A(x)\le\mu_B(x)\le\mu_C(x)\}$, $X_2=\{x/x\in X,\mu_A(x)\le\mu_C(x)<\mu_B(x)\}$,

$X_3=\{x/x\in X,\mu_B(x)<\mu_A(x)\le\mu_C(x)\}$, $X_4=\{x/x\in X,\mu_B(x)\le\mu_C(x)<\mu_A(x)\}$,

$X_5=\{x/x\in X,\mu_C(x)<\mu_A(x)\le\mu_B(x)\}$, $X_6=\{x/x\in X,\mu_C(x)<\mu_B(x)<\mu_A(x)\}$.

In set X_1, $A\cup C=$ Union of A and $C\Leftrightarrow\mu_{A\cup C}(x)=\max\{\mu_A(x),\mu_C(x)\}=\mu_C(x)$;

$B\cup C=$ Union of B and $C\Leftrightarrow\mu_{B\cup C}(x)=\max\{\mu_B(x),\mu_C(x)\}=\mu_C(x)$;

$A\cap C=$ Intersection of A and $C\Leftrightarrow\mu_{A\cap C}(x)=\min\{\mu_A(x),\mu_C(x)\}=\mu_A(x)$;

$B\cap C=$ Intersection of B and $C\Leftrightarrow\mu_{B\cap C}(x)=\min\{\mu_B(x),\mu_C(x)\}=\mu_B(x)$;

Now, $K(A\cup C,B\cup C)$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_{A\cup C}(x_i)\ \mu_{B\cup C}(x_i)}+\sqrt{\left(1-\mu_{A\cup C}(x_i)\right)\left(1-\mu_{B\cup C}(x_i)\right)}\right]}{2}\right)$$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X}\left[\sqrt{\mu_C(x_i)\ \mu_C(x_i)}+\sqrt{\left(1-\mu_C(x_i)\right)\left(1-\mu_C(x_i)\right)}\right]}{2}\right)$$

$=K(C,C)=0$

Also, $K(A\cap C,B\cap C)$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_{A\cap C}(x_i)\ \mu_{B\cap C}(x_i)}+\sqrt{\left(1-\mu_{A\cap C}(x_i)\right)\left(1-\mu_{B\cap C}(x_i)\right)}\right]}{2}\right)$$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_1}^{n}\left[\sqrt{\mu_A(x_i)\ \mu_B(x_i)}+\sqrt{\left(1-\mu_A(x_i)\right)\left(1-\mu_B(x_i)\right)}\right]}{2}\right)$$

$=K(A,B)$

So, $\max\{K(A\cup C,B\cup C),K(A\cap C,B\cap C)\}=K(A,B)$.

Similarly, in the sets X_2,X_3,X_4,X_5,X_6, we have

$\max\{K(A\cup C,B\cup C),K(A\cap C,B\cap C)\}\le K(A,B)$.

Thus, $\max\{K(A\cup C,B\cup C),K(A\cap C,B\cap C)\}\le K(A,B)$ for all $A,B,C\in FS(X)$.

Hence, $K(A,B)$ is valid measure of fuzzy directed divergence.

3.1. Properties of New Fuzzy Divergence Measure $K(A,B)$

In this section, we provide some more properties of the new fuzzy divergence measure in the following theorems. While proving these theorems, we consider the separation of X into two parts X_1 and X_2 as:

$X_1=\{x/x\in X,\mu_A(x_i)\ge\mu_B(x_i)\}$

and

$X_2 = \{x/x \in X, \mu_A(x_i) < \mu_B(x_i)\}.$

In set X_1, $A \cup B =$ Union of A and $B \Leftrightarrow \mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_A(x)$;

$A \cap B =$ Intersection of A and $B \Leftrightarrow \mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_B(x)$.

In set X_2, $A \cup B =$ Union of A and $B \Leftrightarrow \mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_B(x)$;

$A \cap B =$ Intersection of A and $B \Leftrightarrow \mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_A(x)$.

Theorem 3.2: (a) $K(A\cup B, A\cap B) = K(A,B)$

(b) $K(A\cup B, A) + K(A\cap B, A) = K(A,B)$

(c) $K(A\cup B, C) \le K(A,C) + K(B,C)$

(d) $K(A\cap B, C) \le K(A,C) + K(B,C)$

Proof: (a) First, let $K(A\cup B, A\cap B)$

$$= -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_{A\cup B}(x_i)\,\mu_{A\cap B}(x_i)} + \sqrt{(1-\mu_{A\cup B}(x_i))(1-\mu_{A\cap B}(x_i))}\right]}{2}\right)$$

$$= -\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_1}\left[\sqrt{\mu_A(x_i)\,\mu_B(x_i)} + \sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}\right]}{2}\right)$$

$$-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_2}\left[\sqrt{\mu_B(x_i)\,\mu_A(x_i)} + \sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}\right]}{2}\right)$$

$$= -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_A(x_i)\,\mu_B(x_i)} + \sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}\right]}{2}\right)$$

$= K(A,B)$

Hence the result is proved.

(b) $K(A\cup B, A) + K(A\cap B, A)$

$$= -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_{A\cup B}(x_i)\,\mu_A(x_i)} + \sqrt{(1-\mu_{A\cup B}(x_i))(1-\mu_A(x_i))}\right]}{2}\right)$$

$$-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_{A\cap B}(x_i)\,\mu_A(x_i)} + \sqrt{(1-\mu_{A\cap B}(x_i))(1-\mu_A(x_i))}\right]}{2}\right)$$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_1}\left[\sqrt{\mu_A(x_i)\ \mu_A(x_i)}+\sqrt{(1-\mu_A(x_i))(1-\mu_A(x_i))}\right]}{2}\right)$$

$$-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_2}\left[\sqrt{\mu_B(x_i)\ \mu_A(x_i)}+\sqrt{(1-\mu_B(x_i))(1-\mu_A(x_i))}\right]}{2}\right)$$

$$-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_1}\left[\sqrt{\mu_B(x_i)\ \mu_A(x_i)}+\sqrt{(1-\mu_B(x_i))(1-\mu_A(x_i))}\right]}{2}\right)$$

$$-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_2}\left[\sqrt{\mu_A(x_i)\ \mu_A(x_i)}+\sqrt{(1-\mu_A(x_i))(1-\mu_A(x_i))}\right]}{2}\right)$$

$$=K(B,A)$$

Thus, $K(A\cup B,A)+K(A\cap B,A)=K(B,A)=K(A,B)$.

(c) $K(A,C)+K(B,C)-K(A\cup B,C)$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_A(x_i)\ \mu_C(x_i)}+\sqrt{(1-\mu_A(x_i))(1-\mu_C(x_i))}\right]}{2}\right)$$

$$-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_B(x_i)\ \mu_C(x_i)}+\sqrt{(1-\mu_B(x_i))(1-\mu_C(x_i))}\right]}{2}\right)$$

$$+\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_{A\cup B}(x_i)\ \mu_C(x_i)}+\sqrt{(1-\mu_{A\cup B}(x_i))(1-\mu_C(x_i))}\right]}{2}\right)$$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_2}\left[\sqrt{\mu_A(x_i)\ \mu_C(x_i)}+\sqrt{(1-\mu_A(x_i))(1-\mu_C(x_i))}\right]}{2}\right)$$

$$-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_1}\left[\sqrt{\mu_B(x_i)\ \mu_C(x_i)}+\sqrt{\left(1-\mu_B(x_i)\right)\left(1-\mu_C(x_i)\right)}\right]}{2}\right)$$

≥ 0

This proves part (c).

Similarly, we can prove part (d).

Theorem 3.3: For $A,B,C\in FS(X)$,

(a) $K(A\cup B,C)+K(A\cap B,C)=K(A,C)+K(B,C)$

(b) $K(A,A\cap B)=K(B,A\cup B)$

(c) $K(A,A\cup B)=K(B,A\cap B)$

Proof: $K(A\cup B,C)+K(A\cap B,C)$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_{A\cup B}(x_i)\ \mu_C(x_i)}+\sqrt{\left(1-\mu_{A\cup B}(x_i)\right)\left(1-\mu_C(x_i)\right)}\right]}{2}\right)$$

$$-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_{A\cap B}(x_i)\ \mu_C(x_i)}+\sqrt{\left(1-\mu_{A\cap B}(x_i)\right)\left(1-\mu_C(x_i)\right)}\right]}{2}\right)$$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_1}\left[\sqrt{\mu_A(x_i)\ \mu_C(x_i)}+\sqrt{\left(1-\mu_A(x_i)\right)\left(1-\mu_C(x_i)\right)}\right]}{2}\right)$$

$$-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_2}\left[\sqrt{\mu_B(x_i)\ \mu_C(x_i)}+\sqrt{\left(1-\mu_B(x_i)\right)\left(1-\mu_C(x_i)\right)}\right]}{2}\right)$$

$$-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_1}\left[\sqrt{\mu_B(x_i)\ \mu_C(x_i)}+\sqrt{\left(1-\mu_B(x_i)\right)\left(1-\mu_C(x_i)\right)}\right]}{2}\right)$$

$$-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_2}\left[\sqrt{\mu_A(x_i)\ \mu_C(x_i)}+\sqrt{\left(1-\mu_A(x_i)\right)\left(1-\mu_C(x_i)\right)}\right]}{2}\right)$$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_A(x_i)\ \mu_C(x_i)}+\sqrt{\left(1-\mu_A(x_i)\right)\left(1-\mu_C(x_i)\right)}\right]}{2}\right)$$

$$-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_B(x_i)\ \mu_C(x_i)}+\sqrt{\left(1-\mu_B(x_i)\right)\left(1-\mu_C(x_i)\right)}\right]}{2}\right)$$

$$=K(A,C)+K(B,C)$$

(b) $K(A, A\cap B)$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_A(x_i)\ \mu_{A\cap B}(x_i)}+\sqrt{\left(1-\mu_A(x_i)\right)\left(1-\mu_{A\cap B}(x_i)\right)}\right]}{2}\right)$$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_1}\left[\sqrt{\mu_A(x_i)\ \mu_B(x_i)}+\sqrt{\left(1-\mu_A(x_i)\right)\left(1-\mu_B(x_i)\right)}\right]}{2}\right)$$

Now $K(B;\ A\cup B)=-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_B(x_i)\ \mu_{A\cup B}(x_i)}+\sqrt{\left(1-\mu_B(x_i)\right)\left(1-\mu_{A\cup B}(x_i)\right)}\right]}{2}\right)$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_2}\left[\sqrt{\mu_B(x_i)\ \mu_A(x_i)}+\sqrt{\left(1-\mu_B(x_i)\right)\left(1-\mu_A(x_i)\right)}\right]}{2}\right)=K(A, A\cap B)$$

Hence the result holds.

(c) $K(A, A\cup B)=-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_A(x_i)\ \mu_{A\cup B}(x_i)}+\sqrt{\left(1-\mu_A(x_i)\right)\left(1-\mu_{A\cup B}(x_i)\right)}\right]}{2}\right)$

$$=-\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_1}\left[\sqrt{\mu_A(x_i)\ \mu_B(x_i)}+\sqrt{\left(1-\mu_A(x_i)\right)\left(1-\mu_B(x_i)\right)}\right]}{2}\right)$$

Now $K(B, A\cap B)=-\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_B(x_i)\ \mu_{A\cap B}(x_i)}+\sqrt{\left(1-\mu_B(x_i)\right)\left(1-\mu_{A\cap B}(x_i)\right)}\right]}{2}\right)$

$$= -\log\left(\frac{1+\frac{1}{n}\sum_{x\in X_2}\left[\sqrt{\mu_B(x_i)\ \mu_A(x_i)}+\sqrt{\left(1-\mu_B(x_i)\right)\left(1-\mu_A(x_i)\right)}\right]}{2}\right) = K(A, A\cap B)$$

Hence the result holds.

Theorem 3.4. (a) $K\left(A,\overline{A}\right)=K\left(\overline{A},A\right)$.

(b) $K\left(\overline{A},\overline{B}\right)=K(A,B)$.

(c) $K\left(A,\overline{B}\right)=K\left(\overline{A},B\right)$.

(d) $K(A,B)+K\left(\overline{A},B\right)=K\left(\overline{A},\overline{B}\right)+K\left(A,\overline{B}\right)$.

Proof: (a) By using equation (3.1), we have

$$K\left(A,\overline{A}\right) = -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_A(x_i)\ \mu_{\overline{A}}(x_i)}+\sqrt{\left(1-\mu_A(x_i)\right)\left(1-\mu_{\overline{A}}(x_i)\right)}\right]}{2}\right)$$

$$= -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_A(x_i)\left(1-\mu_A(x_i)\right)}+\sqrt{\left(1-\mu_A(x_i)\right)\mu_A(x_i)}\right]}{2}\right)$$

Now, again from equation (3.1), we have

$$K\left(\overline{A},A\right) = -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_{\overline{A}}(x_i)\ \mu_A(x_i)}+\sqrt{\left(1-\mu_{\overline{A}}(x_i)\right)\left(1-\mu_A(x_i)\right)}\right]}{2}\right)$$

$$= -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\left(1-\mu_A(x_i)\right)\mu_A(x_i)}+\sqrt{\mu_A(x_i)\left(1-\mu_A(x_i)\right)}\right]}{2}\right) = K\left(A,\overline{A}\right).$$

(b) $$K\left(\overline{A},\overline{B}\right) = -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_{\overline{A}}(x_i)\ \mu_{\overline{B}}(x_i)}+\sqrt{\left(1-\mu_{\overline{A}}(x_i)\right)\left(1-\mu_{\overline{B}}(x_i)\right)}\right]}{2}\right)$$

$$= -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\left(1-\mu_A(x_i)\right)\left(1-\mu_B(x_i)\right)}+\sqrt{\mu_A(x_i)\ \mu_B(x_i)}\right]}{2}\right) = K(A,B)$$

(c) $$K\left(A,\overline{B}\right) = -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_A(x_i)\ \mu_{\overline{B}}(x_i)}+\sqrt{\left(1-\mu_A(x_i)\right)\left(1-\mu_{\overline{B}}(x_i)\right)}\right]}{2}\right)$$

$$= -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_A(x_i)\ \left(1-\mu_B(x_i)\right)}+\sqrt{\left(1-\mu_A(x_i)\right)\ \mu_B(x_i)}\right]}{2}\right)$$

Thus, $K\left(\overline{A},B\right) = -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\mu_{\overline{A}}(x_i)\ \mu_B(x_i)}+\sqrt{\left(1-\mu_{\overline{A}}(x_i)\right)\left(1-\mu_B(x_i)\right)}\right]}{2}\right)$

$$= -\log\left(\frac{1+\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{\left(1-\mu_A(x_i)\right)\ \mu_B(x_i)}+\sqrt{\mu_A(x_i)\left(1-\mu_B(x_i)\right)}\right]}{2}\right) = K\left(A,\overline{B}\right)$$

(d) It follows from (b) and (c).
This completes the proof.

In the sequel, we have also defined the intuitionistic fuzzy cross-entropy corresponding to fuzzy divergence measure introduced above. Let A and B be two IFSs in the universe of discourse $X=\{x_1,x_2,...,x_n\}$. Then, the intuitionistic fuzzy cross-entropy between IFSs A and B can be defined as

$$D(A,B) = -\log_2\frac{\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\mu_A(x_i)+1-\nu_A(x_i)}{2}\right)\left(\frac{\mu_B(x_i)+1-\nu_B(x_i)}{2}\right)} + \sqrt{\left(\frac{\nu_A(x_i)+1-\mu_A(x_i)}{2}\right)\left(\frac{\nu_B(x_i)+1-\mu_B(x_i)}{2}\right)}\right)\right)}{2}$$

which also indicates discrimination degree of the intuitionistic fuzzy set A from B. This measure also satisfied axioms provided by Couso et. al (2000) defined in section 2. In the next section, we have defined cross-entropy of IVIFSs.

IV. Fuzzy Cross-entropy of Interval valued Intuitionistic Fuzzy sets (IVIFSs)

Let A and B be two IVIFSs in the universe of discourse $X=\{x_1,x_2,...,x_n\}$. IVIFS can be transformed into a fuzzy set to structure a fuzzy cross-entropy of IVIFS by means of

$$\mu_A^F(x_i) = \frac{\mu_{AL}(x_i)+\mu_{AU}(x_i)+2-\nu_{AL}(x_i)-\nu_{AU}(x_i)}{4}$$

Then, by an analogous manner to Zhang and Jiang (2008) and Ye (2009), an IVIFS cross-entropy between IVIFSs A and B can be defined as

$$D(A,B) = -\log_2\frac{\left(1+\frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\left(\frac{\mu_{AL}(x_i)+\mu_{AU}(x_i)+2-\nu_{AL}(x_i)-\nu_{AU}(x_i)}{4}\right)\left(\frac{\mu_{BL}(x_i)+\mu_{BU}(x_i)+2-\nu_{BL}(x_i)-\nu_{BU}(x_i)}{4}\right)} + \sqrt{\left(\frac{\nu_{AL}(x_i)+\nu_{AU}(x_i)+2-\mu_{AL}(x_i)-\mu_{AU}(x_i)}{4}\right)\left(\frac{\nu_{BL}(x_i)+\nu_{BU}(x_i)+2-\mu_{BL}(x_i)-\mu_{BU}(x_i)}{4}\right)}\right)\right)}{2} \quad (4.1)$$

Similarly, we can prove Couso et al. (2000) axioms for the above fuzzy cross-entropy of IVIFSs.

4.1. Optimal decision making method based on the weights of alternatives

In this section, we present an optimal handling method for fuzzy decision making problems based on the weights of alternatives by means of the IVIFS cross entropy.

Let $A=\{A_1, A_2,...,A_m\}$ be a set of alternatives and let $C=\{C_1, C_2,...,C_n\}$ be a set of criteria. Assume that the weight of the criterion $C_j\ (j=1,2,...,n)$, entered by the decision maker, is $w_j,\ w_j\in[0,1]$ and $\sum_{j=1}^{n} w_j=1$. In this case, the characteristics of the alternatives A_i is represented by the following IVIFS:

$A_i=\left\{\left\langle C_j, \left[\mu_{A_iL}(C_j),\ \mu_{A_iU}(C_j)\right], \left[\nu_{A_iL}(C_j),\ \nu_{A_iU}(C_j)\right]\right\rangle \mid C_j\in C\right\}$ and $j=1, 2, 3,...,n$,

where $0\le\mu_{A_iU}(C_j)+\nu_{A_iU}(C_j)\le 1,\quad \mu_{A_iL}(C_j)\ge 0,\ \nu_{A_iL}(C_j)\ge 0,\ j=1,2,...,n$ and $i=1,2,...,m.$ The IVIFS value that is the pair of intervals $\mu_{A_i}(C_j)=[a_{ij},b_{ij}],\ \nu_{A_i}(C_j)=[c_{ij},d_{ij}]$ for $C_j\in C$ is denoted by $\alpha_{ij}=\left([a_{ij},b_{ij}],[c_{ij},d_{ij}]\right)$ for convenience. Here, the interval-valued intuitionistic fuzzy value is usually elicited from the evaluated score to which the alternative A_i satisfies the criterion C_j by means of a score law and data processing or from appropriate membership functions in practice. Therefore, we can elicit a decision matrix $D=(\alpha_{ij})_{m\times n}$.

The aggregating interval-valued intuitionistic fuzzy number α_i for $A_i\,(i=1,2,...,m)$ is $\alpha_i=\left([a_i,b_i],[c_i,d_i]\right)=F_{iw}(\alpha_{i1},...,\alpha_{in})$ or $\alpha_i=\left([a_i,b_i],[c_i,d_i]\right)=G_{iw}(\alpha_{i1},...,\alpha_{in})$, which is obtained by applying equation (2.2) or equation (2.3) according to each row in the decision matrix $D=(\alpha_{ij})_{m\times n}$.

In multi-criteria decision making environments, the concepts of ideal and anti-ideal points have been used to help identify the best alternative in the decision set. Although the ideal alternative does not exist in real world, it does provide a useful theoretical construct against which to evaluate alternatives. Here, we define the ideal and anti-ideal alternatives denoted by the following IVIFSs as

$$A^+=\left\{\left\langle C_j,\alpha_j^+(C_j)\right\rangle \middle| C_j\in C\right\}=\left\{\left\langle C_j,[1,1],[0,0]\right\rangle \middle| C_j\in C\right\},$$

$$A^-=\left\{\left\langle C_j,\alpha_j^-(C_j)\right\rangle \middle| C_j\in C\right\}=\left\{\left\langle C_j,[0,0],[1,1]\right\rangle \middle| C_j\in C\right\}.$$

Then, by applying equation (4.1) a symmetric discrimination information measure (an ideal information measure) between an alternative A_i and the ideal alternative A^+ as $D^+(A_i,A^+)$ & $D^-(A_i,A^-)$.

Let variable u_i denote the global evaluation for alternative A_i, by which the ranking order of all alternatives can be determined. Here, the variable u_i can be interpreted as the weight for the ideal information measure, which describes the difference between each alternative and the ideal alternative. To solve the optimal evaluation of the weight u_i, we make use of the following objective function constructed by Fu, 2008:

$$f(u_i)=u_i^2\left[D^+(A_i,A^+)\right]^2+(1-u_i)^2\left[D^-(A_i,A^-)\right]^2 \tag{4.2}$$

To obtain the optimal evaluation of the weight u_i, we put $\frac{df(u_i)}{du_i}=0.$ Then, we have the following result:

$$u_i=\frac{\left[D^-(A_i,A^-)\right]^2}{\left[D^+(A_i,A^+)\right]^2+\left[D^-(A_i,A^-)\right]^2} \tag{4.3}$$

which provides the global evaluation for each alternative regarding all criteria. From the equation (4.3), the smaller the weight u_i, the better the alternative A_i. Through the weight of each alternative, the best alternative can easily be identified and the ranking order of all alternatives can be determined as well.

V. A Numerical Example

In order to demonstrate the applicability of the proposed method to optimal multi-criteria decision making, we consider below an investment company decision making problem.

Let us suppose that an investment company wants to invest certain amount of money in the best option out of five options: A software company A_1, a pharmaceutical company A_2, a textile company A_3, an automobile company A_4 and a air conditioner company A_5. The investment company needs to take a decision according to the following six criteria: (1) C_1 is the risk analysis (2) C_2 is the growth analysis (3) C_3 is the social-political impact analysis (4) C_4 is the environmental impact analysis (5) C_5 is the level of technology (6) C_6 is service and the criterion weight is $W=(0.20,0.10,0.25,0.10,0.15,0.20)$. The five possible options $A_i\ (i=1,\ 2,\ 3,...,m)$ are to be evaluated by the decision maker under the above six criteria in the following form:

$$\begin{array}{ccccccc}
 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\
A_1 & \begin{pmatrix}[0.5,0.6],\\ [0.2,0.3]\end{pmatrix} & \begin{pmatrix}[0.6,0.7],\\ [0.2,0.3]\end{pmatrix} & \begin{pmatrix}[0.4,0.5],\\ [0.2,0.4]\end{pmatrix} & \begin{pmatrix}[0.7,0.8],\\ [0.1,0.2]\end{pmatrix} & \begin{pmatrix}[0.1,0.3],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.5,0.7],\\ [0.2,0.3]\end{pmatrix} \\
A_2 & \begin{pmatrix}[0.3,0.6],\\ [0.3,0.4]\end{pmatrix} & \begin{pmatrix}[0.1,0.3],\\ [0.7,0.8]\end{pmatrix} & \begin{pmatrix}[0.6,0.7],\\ [0.2,0.3]\end{pmatrix} & \begin{pmatrix}[0.6,0.7],\\ [0.1,0.2]\end{pmatrix} & \begin{pmatrix}[0.3,0.4],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.4,0.7],\\ [0.1,0.2]\end{pmatrix} \\
A_3 & \begin{pmatrix}[0.4,0.5],\\ [0.7,0.8]\end{pmatrix} & \begin{pmatrix}[0.5,0.7],\\ [0.1,0.3]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.3,0.4]\end{pmatrix} & \begin{pmatrix}[0.6,0.7],\\ [0.1,0.3]\end{pmatrix} & \begin{pmatrix}[0.4,0.5],\\ [0.3,0.4]\end{pmatrix} & \begin{pmatrix}[0.3,0.5],\\ [0.1,0.3]\end{pmatrix} \\
A_4 & \begin{pmatrix}[0.6,0.7],\\ [0.2,0.3]\end{pmatrix} & \begin{pmatrix}[0.3,0.4],\\ [0.3,0.5]\end{pmatrix} & \begin{pmatrix}[0.7,0.8],\\ [0.1,0.2]\end{pmatrix} & \begin{pmatrix}[0.3,0.4],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.1,0.3]\end{pmatrix} & \begin{pmatrix}[0.7,0.8],\\ [0.3,0.4]\end{pmatrix} \\
A_5 & \begin{pmatrix}[0.5,0.6],\\ [0.3,0.5]\end{pmatrix} & \begin{pmatrix}[0.4,0.6],\\ [0.2,0.4]\end{pmatrix} & \begin{pmatrix}[0.6,0.7],\\ [0.1,0.3]\end{pmatrix} & \begin{pmatrix}[0.6,0.8],\\ [0.1,0.2]\end{pmatrix} & \begin{pmatrix}[0.6,0.7],\\ [0.2,0.3]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.2,0.4]\end{pmatrix}
\end{array}$$

By using equation (2.2) we can obtain the weighted arithmetic average value (aggregating interval- valued intuitionistic fuzzy value) α_i for $A_i\ (i=1,2,...,5)$:

$\alpha_1=([0.4688,0.6063],[0.2141,0.3435])$, $\alpha_2=([0.4278,0.6162],[0.2291,0.3444])$

$\alpha_3=([0.4426,0.5730],[0.2290,0.4095])$, $\alpha_4=([0.5935,0.7002],[0.1876,0.3239])$

and $\alpha_5=([0.5445,0.6673],[0.1702,0.3478])$

Similarly, the weighted geometric average value is also calculated by applying equation (2.3)

$\alpha_1=([0.3912,0.5569],[0.2457,0.3724])$, $\alpha_2=([0.3629,0.5734],[0.3182,0.4269])$

$\alpha_3=([0.4252,0.5597],[0.3466,0.4877])$, $\alpha_4=([0.5447,0.6494],[0.2313,0.3584])$

and $\alpha_5=([0.5356,0.6568],[0.1883,0.3667])$

By using equation (4.1), we can compute

$D_1^+(A_1,A^+)=0.0086,\ D_2^+(A_2,A^+)=0.0153,\ D_3^+(A_3,A^+)=0.0057,\ D_4^+(A_4,A^+)=0.0094,$

and $D_5^+(A_5,A^+)=0.0022$;

$D_1^-\left(A_1, A^-\right)=0.0093,\ D_2^-\left(A_2, A^-\right)=0.0140,\ D_3^-\left(A_3, A^-\right)=0.0067,\ D_4^-\left(A_4, A^-\right)=0.0079$ and $D_5^-\left(A_5, A^-\right)=0.0022$.

By applying equation (4.3), we have

$u_1=0.5375, u_2=0.8581, u_3=0.5785$ and $u_4=0.4103$.

Therefore, the ranking order of the four alternatives is A_4, A_5, A_1, A_3 and A_2, obviously, amongst them A_4 is the best alternative.

VI. Conclusions

In this paper, we have proposed a fuzzy divergence measure for interval-valued intuitionistic fuzzy sets. An optimal decision making method based on the weights of alternatives has been established by means of the IVIFS cross entropy. Through the weight of each alternative, the best alternative can be easily identified and the ranking order of all alternatives can be determined as well.

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