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Certain Subclasses of Analytic Multivalent Functions using Generalized Differential Operator

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Abstract : *In this paper, we define a new subclass of analytic multivalent functions usinggeneralized Salagean differential operator. Coefficient inequalities, sufficient condition, distortion theorems,radii of starlikeness, convexity and close - to - convexity results are obtained.*
***Keywords** - Coefficient inequalities, Hadamard product , Multivalent functions, , Salagean operator.*

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I. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \geq 0, \tag{1.1}$$

which are analytic in the open disc $\mathrm{U} = \mathrm{U}_1$.

Let S be the subclass of A consisting of functions which are univalent in U. We denote by S^*, C, K and C^* the familiar subclasses of A consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in U. Our favorite references of the field are [1, 2, 3] which covers most of the topics in a lucid and economical style. The concept of starlike functions and convex functions were further extended as follows:

$$\mathrm{S}^*(\alpha) = \left\{ f \in \mathrm{A} : Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \mathrm{U} \right\}, \tag{1.2}$$

$$\mathrm{C}(\alpha) = \left\{ f \in \mathrm{A} : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathrm{U} \right\}. \tag{1.3}$$

We note that

$$f \in \mathrm{C}(\alpha) \Leftrightarrow zf' \in \mathrm{S}^*(\alpha), \tag{1.4}$$

where $\mathrm{S}^*(\alpha)$ and $\mathrm{C}(\alpha)$ are respectively, the classes of starlike of order α and convex of order α in U. (see Robertson [4]).

Similarly, close-to convex functions and quasi-convex functions were further extended as follows:

$$\mathrm{K}(\alpha, \beta) = \left\{ f \in \mathrm{A} : Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha,\ g \in \mathrm{S}^*(\beta),\ z \in \mathrm{U} \right\}, \tag{1.5}$$

Let A_p be the class of functions analytic in the open unit disc $\mathrm{U} = \{z : |z| < 1\}$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \geq 1) \tag{1.6}$$

and let $\mathrm{A} = \mathrm{A}_1$.

A function $f(z) \in \mathrm{A}_p$ is said to be k - uniformly p - valent starlike of order $\delta(-p < \delta < p)$, $k \geq 0$ and $z \in \mathrm{U}$ denoted by $k - UST(p, \delta)$, if and only if

$$Re\left\{\frac{zf'(z)}{f(z)} - \delta\right\} \geq k\left|\frac{zf'(z)}{f(z)} - p\right|$$

A function $f(z)\in A_p$ is said to be k - uniformly p - valent convex of order $\delta(-p<\delta<p)$, $k\geq 0$ and $z\in U$ denoted by $k-UCV(p,\delta)$, if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}-\delta\right\}\geq k\left|1+\frac{zf''(z)}{f'(z)}-p\right|$$

For the functions $f(z)$ of the form (1.6) and $g(z)=z^p+\sum_{k=1}^{\infty}b_{k+p}z^{k+p}$, the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}b_{n+p}z^{n+p}.$$

Let $f(z)$ and $g(z)$ be analytic in U. Then we say that the function $f(z)$ is subordinate to $g(z)$ in U, if there exists an analytic function $w(z)$ in U such that $|w(z)|<|z|$ and $f(z)=g(w(z))$, denoted by $f(z)\prec g(z)$. If $g(z)$ is univalent in U, then the subordination is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

Differentiating both sides of (1.6), q times with respect to z

$$f^{(q)}(z)=\frac{p!}{(p-q)!}z^{p-q}+\sum_{k=1}^{\infty}\frac{(k+p)!}{(k+p-q)!}a_{k+p}z^{k+p-q}, \tag{1.7}$$

$$\left(p\geq 1; q\in \mathbb{N}_0=\mathbb{N}\cup 0; p>q\right).$$

We show by $A_p(q)$ the class of functions of the form (1.7) which are analytic and p - valent in U.

Using the Salagean derivative operator [5], recently Halit Orhan [6] defined the function class as follows:

$$D^n f^{(q)}(z)=\frac{p!}{(p-q)!}z^{p-q}+\sum_{k=1}^{\infty}\frac{(k+p-q)^n(k+p)!}{(p-q)^n(k+p-q)!}a_{k+p}z^{k+p-q}. \tag{1.8}$$

Let T_p denote the subclass of $f\in A_p$ consisting of functions of the form

$$f(z)=z^p-\sum_{k=1}^{\infty}a_{k+p}z^{k+p}\quad \left(p\geq 1 \quad and \quad a_{k+p}\geq 0\right), \tag{1.9}$$

which are p - valent in U.

II. DEFINITIONS AND PRELIMINARIES

Definition 2.1: For any non-zero complex number λ, $0\leq\alpha<1$ and $\beta\geq 0$, a function $f\in A_p$ is said to be in the class $A_n^{p,q}(\beta,\lambda,\alpha)$ if and only if it satisfies the following condition:

$$Re\left\{\frac{\lambda D^{n+1}f^{(q)}(z)}{D^n f^{(q)}(z)}-(\lambda-1)\right\}>\beta\left|\frac{\lambda D^{n+1}f^{(q)}(z)}{D^n f^{(q)}(z)}-\lambda\right|+\alpha. \tag{2.1}$$

Remark 2.1**:** By specializing the parameters we get the following well-known results:

(1) If we let $q=0$, then the above inequality (2.1), reduces to

$$Re\left\{\frac{\lambda D^{n+1}f(z)}{D^n f(z)}-(\lambda-1)\right\}>\beta\left|\frac{\lambda D^{n+1}f(z)}{D^n f(z)}-\lambda\right|+\alpha.$$

(2) If we choose $q=0$, $\lambda=2$, $p=1$ and $n=1$ in (2.1) then the function class $A_n^{p,q}(\beta,\lambda,\alpha)$ reduces to the class $\beta-UCV(\alpha)$ of functions $f\in A$ satisfying the condition

$$Re\left\{\frac{\left(zf'(z)\right)'}{f'(z)}\right\} > \beta\left|\frac{\left(zf'(z)\right)'}{f'(z)}\right| + \alpha.$$

The class $\beta - UCV(\alpha)$ was introduced and studied by R. Bharathi et. al. [7].

Also we define the class $\mathrm{T}_n^{p,q}(\beta,\lambda,\alpha)$ by

$$\mathrm{T}_n^{p,q}(\beta,\lambda,\alpha) = A_n^{p,q}(\beta,\lambda,\alpha) \cap \mathrm{T}_p.$$

Motivated by the concept introduced by Halit Orhan [6] and various authors in [8, 9], we introduce the new subclass of analytic p - valent function using Salagean operators. We obtain coefficient estimates, distortion bounds and radii of starlikeness for the above said function classes.

III. SUFFICIENT CONDITION

In this section, we find the sufficient condition for our function class $\mathrm{T}_n^{p,q}(\beta,\lambda,\alpha)$.

Theorem 3.1: A function $f(z)$ defined by (1.9) is in $\mathrm{T}_n^{p,q}(\beta,\lambda,\alpha)$ if and only if

$$\sum_{k=1}^{\infty}\left[(1-\alpha) + \lambda k(1+\beta)(p-q)^{-1}\right] \psi(k) a_{k+p} = \frac{(1-\alpha)p!}{(p-q)!}, \tag{3.1}$$

where $\psi(k) = \dfrac{(k+p-q)^n (k+p)!}{(p-q)^n (k+p-q)!}$.

Proof: By definition $f(z) \in \mathrm{T}_n^{p,q}(\beta,\lambda,\alpha)$ if and only if the condition (2.1) satisfied. Using the fact that

$$Re\{\lambda w - (\lambda - 1)\} = \beta|\lambda(w-1)| + \alpha$$

$$\Leftrightarrow Re\left\{(\lambda w - (\lambda-1))\left(1+\beta e^{i\theta}\right) - \beta e^{i\theta}\right\} > \alpha, \quad -\pi \le \theta < \pi. \tag{3.2}$$

Using (3.2), equation (2.1) can be written as

$$Re\left\{\left(\frac{\lambda D^{n+1} f^{(q)}(z)}{D^n f^{(q)}(z)} - (\lambda-1)\right)\left(1+\beta e^{i\theta}\right) - \beta e^{i\theta}\right\} > \alpha,$$

equivalently,

$$Re\left\{\left(\frac{\lambda D^{n+1} f^{(q)}(z) - (\lambda-1)D^n f^{(q)}(z)}{D^n f^{(q)}(z)}\right)\left(1+\beta e^{i\theta}\right) - \frac{\beta e^{i\theta} D^n f^{(q)}(z)}{D^n f^{(q)}(z)}\right\} > \alpha. \tag{3.3}$$

Let $G(z) = \left[\lambda D^{n+1} f^{(q)}(z) - (\lambda-1)D^n f^{(q)}(z)\right]\left(1+\beta e^{i\theta}\right) - \beta e^{i\theta} D^n f^{(q)}(z)$ and let $H(z) = D^n f^{(q)}(z)$.

Then the above equation is equivalent to

$$|G(z) + (1-\alpha)H(z)| > |G(z) - (1+\alpha)H(z)|, \qquad for \quad 0 \le \alpha < 1.$$

For $G(z)$ and $H(z)$ as above, we have

$$\begin{aligned}|G(z)+(1-\alpha)H(z)| &= \Bigg|\frac{(2-\alpha)p!}{(p-q)!} z^{p-q} - \sum_{k=1}^{\infty}\left[(2-\alpha) + \lambda k(p-q)^{-1}\right]\psi(k) a_{k+p} z^{k+p-q} \\ &\qquad - \lambda\beta e^{i\theta}\sum_{k=1}^{\infty} k(p-q)^{-1}\psi(k) a_{k+p} z^{k+p-q}\Bigg| \\ &\ge \frac{(2-\alpha)p!}{(p-q)!}|z|^{p-q} - \sum_{k=1}^{\infty}\left[(2-\alpha) + \lambda k(1+\beta)(p-q)^{-1}\right]\psi(k) a_{k+p} |z|^{k+p-q}.\end{aligned}$$

Similarly,

$$|G(z) - (1+\alpha)H(z)| < \frac{\alpha p!}{(p-q)!}|z|^{p-q} + \sum_{k=1}^{\infty}\left[\alpha - \lambda k(1+\beta)(p-q)^{-1}\right] \psi(k) a_{k+p} |z|^{k+p-q}.$$

Now

$$|G(z)+(1-\alpha)H(z)| - |G(z)-(1+\alpha)H(z)|$$

$$\geq \frac{2(1-\alpha)p!}{(p-q)!}|z|^{p-q} - 2\sum_{k=1}^{\infty}\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right]\psi(k)a_{k+p}|z|^{k+p-q} \geq 0.$$

Or equivalently,

$$\sum_{k=1}^{\infty}\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right]\psi(k)a_{k+p} \leq \frac{(1-\alpha)p!}{(p-q)!}$$

which is the required assertion of the Theorem 3.1.
On the otherhand, we must have

$$Re\left\{\left(\frac{\lambda D^{n+1}f^{(q)}(z)}{D^n f^{(q)}(z)}-(\lambda-1)\right)\left(1+\beta e^{i\theta}\right)-\beta e^{i\theta}\right\} > \alpha, \qquad -\pi \leq \theta < \pi.$$

Upon choosing the values of z on the positive real axis where $0 \leq |z| = r < 1$, the above inequality reduces to

$$Re\left\{\frac{\frac{(1-\alpha)p!}{(p-q)!}-\sum_{k=1}^{\infty}\frac{\lambda k+(1-\alpha)}{(p-q)}\psi(k)a_{k+p}r^k - \lambda\beta e^{i\theta}\sum_{k=1}^{\infty}k(p-q)^{-1}\psi(k)a_{k+p}r^k}{\frac{p!}{(p-q)!}-\sum_{k=1}^{\infty}\psi(k)a_{k+p}r^k}\right\} \geq 0.$$

Since $Re(-e^{i\theta}) \geq -\left|e^{i\theta}\right| = -1$, then the above inequality gives

$$Re\left\{\frac{\frac{(1-\alpha)p!}{(p-q)!}-\sum_{k=1}^{\infty}\frac{\lambda k+(1-\alpha)}{(p-q)}\psi(k)a_{k+p}r^k + \lambda\beta e^{i\theta}\sum_{k=1}^{\infty}k(p-q)^{-1}\psi(k)a_{k+p}r^k}{\frac{p!}{(p-q)!}-\sum_{k=1}^{\infty}\psi(k)a_{k+p}r^k}\right\} \geq 0.$$

Letting $r \to 1^-$, we get the desired result. □

Finally, the function $f(z)$ given by

$$f(z) = z^p - \frac{(1-\alpha)p!}{(p-q)!\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right]\psi(k)}z^{p+1} \tag{3.4}$$

is an extremal function for the assertion of the Theorem 3.1.

Corollary 3.2: If $f(z) \in T_n^{p,q}(\beta,\lambda,\alpha)$, then

$$a_{k+p} \leq \frac{(1-\alpha)p!}{(p-q)!\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right]\psi(k)}. \tag{3.5}$$

Equality being attained for the function $f(z)$ given by (3.4).

IV. GROWTH AND DISTORTION THEOREMS

Theorem 4.1: Let the function $f(z)$ defined by (1.9) be in the class $T_n^{p,q}(\beta,\lambda,\alpha)$. Then for $|z| = r$, we have

$$r^p - \frac{(1-\alpha)p!}{(p-q)!\left[(1-\alpha)+\lambda(1+\beta)(p-q)^{-1}\right]\psi(1)}r^{p+1} \leq |f(z)| \leq \tag{4.1}$$

$$r^p + \frac{(1-\alpha)p!}{(p-q)!\left[(1-\alpha)+\lambda(1+\beta)(p-q)^{-1}\right]\psi(1)}r^{p+1}.$$

Proof: Given that $f(z)$ in $T_n^{p,q}(\beta,\lambda,\alpha)$, from the equation (3.1), we have

$$\left[(1-\alpha)+\lambda(1+\beta)(p-q)^{-1}\right]\psi(1)\sum_{k=1}^{\infty}a_{k+p} \le \sum_{k=1}^{\infty}\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right]\psi(k)a_{k+p} \le \frac{(1-\alpha)p!}{(p-q)!} \tag{4.2}$$

which implies

$$\sum_{k=1}^{\infty}a_{k+p} \le \frac{(1-\alpha)p!}{(p-q)!\left[(1-\alpha)+\lambda(1+\beta)(p-q)^{-1}\right]\psi(1)}a_{k+p}. \tag{4.3}$$

Using (1.9) and (4.3), we obtain

$$\begin{aligned}|f(z)| &\le |z|^p+\sum_{k=1}^{\infty}a_{k+p}|z|^{k+p}\\ &\le r^p+\sum_{k=1}^{\infty}a_{k+p}r^{k+p}\\ &\le r^p+\frac{(1-\alpha)p!}{(p-q)!\left[(1-\alpha)+\lambda(1+\beta)(p-q)^{-1}\right]\psi(1)}r^{p+1}.\end{aligned} \tag{4.4}$$

Similarly,

$$|f(z)| \ge r^p-\frac{(1-\alpha)p!}{(p-q)!\left[(1-\alpha)+\lambda(1+\beta)(p-q)^{-1}\right]\psi(1)}r^{p+1}, \tag{4.5}$$

which proves the assertion (4.1). □

Theorem 4.2: Let the function $f(z)$ defined by (1.9) be in the class $\mathrm{T}_n^{p,q}(\beta,\lambda,\alpha)$. Then for $|z|=r$, we have

$$pr^{p-1}-\frac{(1-\alpha)p!(1+p)}{(p-q)!\left[(1-\alpha)+\lambda(1+\beta)(p-q)^{-1}\right]\psi(1)}r^p \le |f'(z)| \le pr^{p-1}+\frac{(1-\alpha)p!(1+p)}{(p-q)!\left[(1-\alpha)+\lambda(1+\beta)(p-q)^{-1}\right]\psi(1)}r^p. \tag{4.6}$$

Proof: We have

$$f'(z)=pz^{p-1}+\sum_{k=1}^{\infty}(k+p)a_{k+p}z^{k+p-1}, \tag{4.7}$$

which implies

$$\begin{aligned}|f'(z)| &\le p|z|^{p-1}+\sum_{k=1}^{\infty}(k+p)a_{k+p}|z|^{k+p-1}\\ &\le pr^{p-1}+\sum_{k=1}^{\infty}(k+p)a_{k+p}r^{k+p-1}\\ &\le pr^{p-1}+\frac{(1-\alpha)p!(1+p)}{(p-q)!\left[(1-\alpha)+\lambda(1+\beta)(p-q)^{-1}\right]\psi(1)}r^p.\end{aligned} \tag{4.8}$$

Similarly, we get,

$$|f'(z)| \ge pr^{p-1}-\frac{(1-\alpha)p!(1+p)}{(p-q)!\left[(1-\alpha)+\lambda(1+\beta)(p-q)^{-1}\right]\psi(1)}r^p. \tag{4.9}$$

This completes the proof of Theorem 4.2. □

If we let $n = 0$ in the above results, we get the following corollary:

Corollary 4.3: If $f \in \mathrm{T}_0^{p,q}\left(\beta,\lambda,\alpha\right)$, then for $|z| = r$,

$$r^p - \frac{(1-\alpha)(p-q)(p-q+1)}{\left[(1-\alpha)(p-q)+\lambda(1+\beta)\right](p+1)} r^{p+1} \le \ |f(z)| \le r^p + \frac{(1-\alpha)(p-q)(p-q+1)}{\left[(1-\alpha)(p-q)+\lambda(1+\beta)\right](p+1)} r^{p+1}$$

and

$$pr^{p-1} - \frac{(1-\alpha)(p-q)(p-q+1)}{\left[(1-\alpha)(p-q)+\lambda(1+\beta)\right]} r^p \le \ |f'(z)| \le pr^{p-1} + \frac{(1-\alpha)(p-q)(p-q+1)}{\left[(1-\alpha)(p-q)+\lambda(1+\beta)\right]} r^p. \quad \square$$

If we let $q = 0$ in the above corollary, we get

Corollary 4.4: If $f \in \mathrm{T}_0^{p,0}\left(\beta,\lambda,\alpha\right)$, then for $|z| = r$,

$$r^p - \frac{(1-\alpha)p}{\left[(1-\alpha)p+\lambda(1+\beta)\right]} r^{p+1} \le |f(z)| \le r^p + \frac{(1-\alpha)p}{\left[(1-\alpha)p+\lambda(1+\beta)\right]} r^{p+1}$$

and

$$pr^{p-1} - \frac{(1-\alpha)p(p+1)}{\left[(1-\alpha)p+\lambda(1+\beta)\right]} r^p \le |f'(z)| \le pr^{p-1} + \frac{(1-\alpha)p(p+1)}{\left[(1-\alpha)p+\lambda(1+\beta)\right]} r^p.$$

V. Radii Of Starlikeness, Convexity And Close-To-Convexity

In this section, radii of close - to- convexity, starlikeness and convexity for the function class $\mathrm{T}_n^{p,q}\left(\beta,\lambda,\alpha\right)$ are discussed.

The real number

$$r_\rho^*(f) = \sup\left\{r > 0 | Re\left(k(z)\right) > \rho \ \ forall \ \ z \in \mathrm{U}_r\right\}$$

is called the radius of starlikeness of order ρ of the function f when $k(z) = \dfrac{zf'(z)}{f(z)}$. Note that $r_\rho^*(f) = r_0^*(f)$ is in fact the largest radius such that the image region $f\left(\mathrm{U}_r^*(f)\right)$ is a starlike domain with respect to the origin. Similar definition is used to define radius of convexity and close to convexity by equivalently replacing $k(z)$ with $1 + \dfrac{zf''(z)}{f'(z)}$ and $\dfrac{f'(z)}{g'(z)}$ respectively. For the study of various radius problems, we refer to [10, 11, 12, 13, and 14].

Theorem 5.1: Let the function $f(z)$ defined by (1.9) be in the class $\mathrm{T}_n^{p,q}\left(\beta,\lambda,\alpha\right)$. Then $f(z)$ is p - valent starlike of order $\sigma(0 \le \sigma < 1)$ in $|z| < R_1$, where

$$R_1 = \inf_{k\ge1}\left\{\frac{\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right](p-q)!\psi(k)}{(1-\alpha)p!} \times \left(\frac{p-\sigma}{k+p-\sigma}\right)\right\}^{\frac{1}{k}} \qquad (z \in \mathrm{U}), \tag{5.1}$$

where, $\psi(k) = \dfrac{\left(k+p-q\right)^n\left(k+p\right)!}{\left(p-q\right)^n\left(k+p-q\right)!}$.

Proof: Given $f \in \mathrm{T}_n^{p,q}\left(\beta,\lambda,\alpha\right)$ and f is starlike of order σ, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \sigma. \tag{5.2}$$

For the left hand side of (5.2), we have

$$\left|\frac{zf'(z)}{f(z)}-p\right|=\left|\frac{pz^{p}-\sum_{k=1}^{\infty}(k+p)a_{k+p}z^{k+p}}{z^{p}-\sum_{k=1}^{\infty}a_{k+p}z^{k+p}}-p\right|\le\frac{\sum_{k=1}^{\infty}ka_{k+p}|z|^{k}}{1-\sum_{k=1}^{\infty}a_{k+p}|z|^{k}}.$$

The last expression is less than $(p-\sigma)$ if

$$\frac{\sum_{k=1}^{\infty}ka_{k+p}|z|^{k}}{1-\sum_{k=1}^{\infty}a_{k+p}|z|^{k}}<p-\sigma,$$

which implies

$$\sum_{k=1}^{\infty}\frac{k+p-\sigma}{p-\sigma}a_{k+p}|z|^{k}<1.$$

Using the fact that, $f\in T_{n}^{p,q}(\beta,\lambda,\alpha)$ if and only if

$$\sum_{k=1}^{\infty}\frac{\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right](p-q)!\psi(k)}{(1-\alpha)p!}a_{k+p}\le 1.$$

We can say that (5.2) is true if

$$\frac{k+p-\sigma}{p-\sigma}|z|^{k}\le\frac{\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right](p-q)!\psi(k)}{(1-\alpha)p!}.$$

$$\Rightarrow|z|^{k}\le\frac{\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right](p-q)!\psi(k)}{(1-\alpha)p!}\left(\frac{p-\sigma}{k+p-\sigma}\right).$$

The last inequality leads us immediately to the disc $|z|<R_1$, where R_1 is given by (5.1). □

Theorem 5.2: Let the function $f(z)$ defined by (1.9) be in the class $T_{n}^{p,q}(\beta,\lambda,\alpha)$. Then $f(z)$ is p - valent close - to - convex of order $\sigma(0\le\sigma<1)$ *in* $|z|<R_2$, where

$$R_{2}=\inf_{k\ge1}\left\{\frac{\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right](p-q)!\psi(k)}{(1-\alpha)p!}\times\left(\frac{p-\sigma}{k+p}\right)\right\}^{\frac{1}{k}}\qquad(z\in U),\qquad(5.3)$$

$$where,\psi(k)=\frac{(k+p-q)^{n}(k+p)!}{(p-q)^{n}(k+p-q)!}.$$

Proof: Given $f\in T_{n}^{p,q}(\beta,\lambda,\alpha)$ and f is p - valent close - to - convex of order σ, we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|<p-\sigma.\qquad(5.4)$$

For the left hand side of (5.4), we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|=\left|\frac{pz^{p-1}-\sum_{k=1}^{\infty}(k+p)a_{k+p}z^{k+p-1}}{z^{p-1}}-p\right|\le\sum_{k=1}^{\infty}(k+p)a_{k+p}|z|^{k}.$$

The last expression is less than $(p-\sigma)$ if

$$\sum_{k=1}^{\infty}(k+p)a_{k+p}|z|^{k}<p-\sigma$$

which implies

$$\sum_{k=1}^{\infty}\frac{k+p}{p-\sigma}a_{k+p}\left|z\right|^{k}<1.$$

Using the fact that, $f\in \mathrm{T}_{n}^{p,q}\left(\beta,\lambda,\alpha\right)$ if and only if

$$\sum_{k=1}^{\infty}\frac{\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right](p-q)!\psi(k)}{(1-\alpha)p!}a_{k+p}\leq 1.$$

We can say that (5.4) is true if

$$\frac{k+p}{p-\sigma}\left|z\right|^{k}\leq\frac{\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right](p-q)!\psi(k)}{(1-\alpha)p!}.$$

$$\Rightarrow\left|z\right|^{k}\leq\frac{\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right](p-q)!\psi(k)}{(1-\alpha)p!}\left(\frac{p-\sigma}{k+p}\right).$$

The last inequality leads us immediately to the disc $\left|z\right|<R_2$, where R_2 is given by (5.3). □

Theorem 5.3: Let the function $f(z)$ defined by (1.9) be in the class $\mathrm{T}_{n}^{p,q}\left(\beta,\lambda,\alpha\right)$. Then $f(z)$ is p - valent convex of order $\sigma(0\leq\sigma<1)$ *in* $\left|z\right|<R_3$, where

$$R_3=\inf_{k\geq 1}\left\{\frac{\left[(1-\alpha)+\lambda k(1+\beta)(p-q)^{-1}\right](p-q)!\psi(k)}{(1-\alpha)p!}\times\left(\frac{p(p-\sigma)}{(k+p)(k+p-\sigma)}\right)\right\}^{\frac{1}{k}}\qquad (z\in \mathrm{U}),\qquad (5.5)$$

$$where,\psi(k)=\frac{\left(k+p-q\right)^{n}\left(k+p\right)!}{\left(p-q\right)^{n}\left(k+p-q\right)!}.$$

Proof: To prove (5.5), it is sufficient to show that

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right|\leq 1-\sigma.$$

The proof is omitted, since we use a similar proof of Theorem 5.2. □

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