K-Even Edge-Graceful Labeling of Some Cycle Related Graphs

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ABSTRACT: In 1985, Lo[6] introduced the notion of edge-graceful graphs. In [4], Gayathri et al., introduced the even edge-graceful graphs. In [8], Sin-Min Lee, Kuo-Jye Chen and Yung-Chin Wang introduced the k-edge-graceful graphs. We introduced k-even edge-graceful graphs. In this paper, we investigate the k-even edge-gracefulness of some cycle related graphs.

KEYWORDS: k-even edge-graceful labeling, k-even edge-graceful graphs. AMS(MOS) subject classification: 05C78.

I. INTRODUCTION

All graphs in this paper are finite, simple and undirected. Terms not defined here are used in the sense of Harary[5]. The symbols V(G) and E(G) will denote the vertex set and edge set of a graph G. The cardinality of the vertex set is called the order of G denoted by p. The cardinality of the edge set is called the size of G denoted by q. A graph with p vertices and q edges is called a (p, q) graph.

In 1985, Lo[6] introduced the notion of edge-graceful graphs. In [4], Gayathri et al., introduced the even edge-graceful graphs and further studied in. In [8], Sin-Min Lee, Kuo-Jye Chen and Yung-Chin Wang introduced the k-edge-graceful graphs. We have introduced k-even edge-graceful graphs.

Definition 1.1:

k-even edge-graceful labeling (k-EEGL) of a (p, q) graph G(V, E) is an injection f from E to {2k − 1, 2k, 2k + 1, ..., 2k + 2q − 1} such that the induced mapping f^* defined on V by f^*(x) = (Σf(xy)) (mod 2s) taken over all edges xy are distinct and even where s = max{p, q} and k is an integer greater than or equal to 1. A graph G that admits k-even edge-graceful labeling is called a k-even edge-graceful graph (K-EEGG).

Remark 1.2:

1-even edge-graceful labeling is an even edge-graceful labeling. The definition of k-edge-graceful and k-even edge-graceful are equivalent to one another in the case of trees.

The edge-gracefulness and even edge-gracefulness of odd order trees are still open. The theory of 1-even edge-graceful is completely different from that of k-even edge-graceful. For example, tree of order 4 is 2-even edge-graceful but not 1-even edge-graceful. In this paper we investigate the k-even edge-gracefulness of some cycle related graphs. Throughout this paper, we assume that k is a positive integer greater than or equal to 1.

2. Prior Results:

1. Theorem: If a (p, q) graph G is k-even edge-graceful with all edges labeled with even numbers and p ≥ q then G is k-edge-graceful.

2. Theorem: If a (p, q) graph G is k-even edge-graceful in which all edges are labeled with even numbers and

\[ p \geq q \text{ then } q(q + 2k - 1) = \frac{p(p + 1)}{2} (\text{mod } p). \]

Further \[ q(q + 2k - 1) = \begin{cases} 0 (\text{mod } p) & \text{if } p \text{ is odd} \\ \frac{p}{2} (\text{mod } p) & \text{if } p \text{ is even} \end{cases} \]
3. **Theorem**: If a \((p, q)\) graph \(G\) is \(k\)-even edge-graceful in which all edges are labeled with even numbers and \(p \geq q\) then \(p = 0, 1\) or \(3 \pmod{4}\).

4. **Theorem**: If a \((p, q)\) graph \(G\) is a \(k\)-even edge-graceful tree of odd order then

\[ k = \frac{p}{2} (l - 1) + 1 \]

where \(l\) is any odd positive integer and hence \(k = 1 \pmod{p}\).

5. **Observation**: We observe that any tree of odd order \(p\) has the sum of the labels congruent to 0 \((\pmod{p})\).

6. **Theorem**: If a \((p, q)\) graph \(G\) is a \(k\)-even edge-graceful tree of even order with \(p = 0 \pmod{4}\) then

\[ k = \frac{p}{4} (2l - 1) + 1 \]

where \(l\) is any positive integer.

2. **MAIN RESULTS**

2.1 **Definition**

Let \(C_n\) denote the cycle of length \(n\). Then the join of \([e]\) with any one vertex of \(C_n\) is denoted by \(C_n \cup [e]\). In this graph, \(p = q = n + 1\).

2.2 **Theorem**

The graph \(C_n \cup [e]\) of even order is \(k\)-even-edge-graceful for all \(k = z \left( \frac{p}{2} \right) \), where \(0 \leq z \leq \frac{p}{2} - 1\), \(n \equiv 1 \pmod{4}\) and \(n \neq 1\).

**Proof**

Let \(\{v_1, v_2, \ldots, v_n, e\}\) be the vertices of \(C_n \cup [e]\), the edges \(e_i = (v_i, v_{i+1})\) for \(1 \leq i \leq n-1\); \(e_n = (v_n, v_1)\) and \(e_{n+1} = (v_2, v)\) (see Figure 1).

\[ \begin{align*}
 v_1 & \quad v_2 & \quad \ldots & \quad v_{n-1} & \quad v_n & \quad v_1 \\
 e_1 & \quad e_2 & \quad \ldots & \quad e_{n-1} & \quad e_n & \quad e_{n+1}
\end{align*} \]

**Figure 1**: \(C_n \cup [e]\) with ordinary labeling

First, we label the edges as follows:

For \(k \geq 1\), \(1 \leq i \leq n\) and \(i\) is odd,

\[ f(e_i) = 2k + i - 2. \]

When \(i\) is even, we label the edges as follows:

For \(k = z \left( \frac{p}{2} \right)\), \(0 \leq z \leq \frac{p - 2}{4}\),
\( f(e_i) = \begin{cases} 
2k + n + i - 2 & \text{for } 1 \leq i < \frac{n - 4z + 7}{2} \\
2k + n + i & \text{for } \frac{n - 4z + 7}{2} \leq i \leq n.
\end{cases} \)

For \( k = \frac{p + 2}{4} \left( \text{mod } \frac{p}{2} \right) \),
\( f(e_i) = 2k + n + i \).

For \( k = \frac{p + 6}{4} \left( \text{mod } \frac{p}{2} \right) \),
\( f(e_i) = 2k + n + i - 2 \).

For \( k = z \left( \text{mod } \frac{p}{2} \right) \), \( \frac{p + 10}{4} \leq z \leq \frac{p}{2} - 1 \),
\( f(e_i) = \begin{cases} 
2k + n + i - 2 & \text{for } 1 \leq i < \frac{3n - 4z + 9}{2} \\
2k + n + i & \text{for } \frac{3n - 4z + 9}{2} \leq i \leq n.
\end{cases} \)

\( f(e_{n+1}) = \begin{cases} 
2k & \text{when } k = 0 \left( \text{mod } p \right) \\
2k + 2n - 2z + 2 & \text{when } k = z \left( \text{mod } p \right) \text{ and } 1 \leq z \leq p - 1.
\end{cases} \)

Then the induced vertex labels are as follows:

**Case 1:** \( k \equiv z \left( \text{mod } \frac{p}{2} \right) \), \( 0 \leq z \leq \frac{p}{4} - 1 \),
\( f^*(v_i) = \begin{cases} 
n + 4z + 2i - 5 & \text{for } 1 \leq i < \frac{n - 4z + 7}{2} \\
4z - n + 2i - 5 & \text{for } \frac{n - 4z + 7}{2} \leq i \leq n.
\end{cases} \)

**Case 2:** \( k = \frac{p + 2}{4} \left( \text{mod } \frac{p}{2} \right) \)
\( f^*(v_i) = 2n ) \); \( f^*(v_i) = 2i - 2 \) \text{ for } 2 \leq i \leq n.\)

**Case 3:** \( k = \frac{p + 6}{4} \left( \text{mod } \frac{p}{2} \right) \)
\( f^*(v_i) = 2i \) \text{ for } 1 \leq i \leq n.\)

**Case 4:** \( k = z \left( \text{mod } \frac{p}{2} \right) \), \( \frac{p + 10}{4} \leq z \leq \frac{p}{2} - 1 \)
\[ f^\ast(v_i) = \begin{cases} 
4z - n + 2i - 7 & \text{for } 1 \leq i < \frac{3n - 4z + 9}{2} \\
4z - 3n + 2i - 7 & \text{for } \frac{3n - 4z + 9}{2} \leq i \leq n. 
\end{cases} \]

For \( k = z \left( \text{mod} \, \frac{p}{2} \right) \), \( 0 \leq z \leq \frac{p}{2} - 1 \),

\[ f^\ast(v_{n+1}) = 0. \]

Therefore, \( f^\ast(V) = \{0, 2, 4, \ldots, 2s - 2\} \), where \( s = \max\{p, q\} = n + 1 \). So, it follows that the vertex labels are all distinct and even. Hence, the graph \( C_n \cup \{e\} \) of even order is \( k \)-even-edge-graceful for all \( k = z \left( \text{mod} \, \frac{p}{2} \right) \), where \( 0 \leq z \leq \frac{p}{2} - 1 \), \( n = 1 \pmod 4 \) and \( n \neq 1 \).

For example, consider the graph \( C_{13} \cup \{e\} \). Here \( n = 13; \)

\( s = \max\{p, q\} = 14; 2s = 28 \). The 21-EEGL of \( C_{13} \cup \{e\} \) is given in Figure 2.

For example, consider the graph \( C_{17} \cup \{e\} \). Here \( n = 17; \)

\( s = \max\{p, q\} = 18; 2s = 36 \). The 5-EEGL of \( C_{17} \cup \{e\} \) is given in Figure 3.
Definition 2.3 [8]
For \( p \geq 4 \), a cycle (of order \( p \)) with one chord is a simple graph obtained from a \( p \)-cycle by adding a chord. Let the \( p \)-cycle be \( v_1v_2 \ldots v_pv_1 \). Without loss of generality, we assume that the chord joins \( v_1 \) with any one \( v_i \), where \( 3 \leq i \leq p - 1 \). This graph is denoted by \( C_p(i) \). For example \( C_5(5) \) means a graph obtained from a \( p \)-cycle by adding a chord between the vertices \( v_1 \) and \( v_5 \). In this graph, \( q = p + 1 \).

Theorem 2.4
The graph \( C_n(i), (n > 4), 3 \leq i \leq n - 1 \), cycle with one chord of odd order is \( k \)-even-edge-graceful for all \( k \equiv z \mod \frac{q}{2} \), where \( 0 \leq z \leq \frac{q}{2} - 1 \).

Proof
Let \( \{v_1, v_2, v_3, \ldots, v_n\} \) be the vertices of \( C_n(i) \), the edges \( e_i = (v_i, v_{i+1}) \) for \( 1 \leq i \leq n - 1 \); \( e_n = (v_n, v_1) \) and \( e_{n+1} = (v_1, v_i) \), \( 3 \leq i \leq n - 1 \) (see Figure 4). The chord connecting the vertex \( v_1 \) with \( v_i \), \( (i \geq 3) \) is shown in Figure 4. For this graph, \( p = n \) and \( q = n + 1 \).

![Figure 4: C_n(i) with ordinary labeling](https://www.ijesi.org)

First, we label the edges as follows:
For \( k \geq 1 \) and \( 1 \leq i \leq n \),

\[
 f(e_i) = \begin{cases} 
 2k + i - 2 & \text{when } i \text{ is odd} \\
 2k + n + i - 2 & \text{when } i \text{ is even}.
\end{cases}
\]

\[
 f(e_{n+1}) = \begin{cases} 
 2k & \text{when } k = 0 \mod q \\
 2k + 2n - 2z + 2 & \text{when } k = z \mod q, 1 \leq z \leq q - 1.
\end{cases}
\]

Then the induced vertex labels are as follows:

**Case 1: \( n = 1 \mod 4 \) and \( n \neq 1 \)**

Case 1: \( k = z \left( \mod \frac{q}{2} \right) \), \( 0 \leq z \leq \frac{q + 2}{4} \)

\[
 f^+(v_i) = \begin{cases} 
 4z + n + 2i - 5 & \text{for } 1 \leq i < \frac{n - 4z + 7}{2} \\
 4z + n + 2i - 7 & \text{for } \frac{n - 4z + 7}{2} \leq i \leq n.
\end{cases}
\]
Case 2: \( k = \frac{q + 6}{4} \left( \text{mod} \frac{q}{2} \right) \)

\[
f^{'\prime}(v_i) = 2i \quad \text{for } 1 \leq i \leq n.
\]

Case 3: \( k = z \left( \text{mod} \frac{q}{2} \right), \frac{q + 10}{4} \leq z \leq \frac{q}{2} - 1 \)

\[
f^{'\prime}(v_i) = \begin{cases} 
4z - n + 2i - 7 & \text{for } 1 \leq i < \frac{3n - 4z + 9}{2} \\
4z - 3n + 2i - 9 & \text{for } \frac{3n - 4z + 9}{2} \leq i \leq n.
\end{cases}
\]

Case II: \( n \equiv 3 \pmod{4} \) and \( n \neq 3 \)

Case 1: \( k = z \left( \text{mod} \frac{q}{2} \right), 0 \leq z \leq \frac{q}{4} \)

\[
f^{'\prime}(v_i) = \begin{cases} 
4z + n + 2i - 5 & \text{for } 1 \leq i < \frac{n - 4z + 7}{2} \\
4z - n + 2i - 7 & \text{for } \frac{n - 4z + 7}{2} \leq i \leq n.
\end{cases}
\]

Case 2: \( k = \frac{q + 4}{4} \left( \text{mod} \frac{q}{2} \right) \)

\[
f^{'\prime}(v_i) = 2i - 2 \quad \text{for } 1 \leq i \leq n.
\]

Case 3: \( k = z \left( \text{mod} \frac{q}{2} \right), \frac{q + 8}{4} \leq z \leq \frac{q}{2} - 1 \)

\[
f^{'\prime}(v_i) = \begin{cases} 
4z - n + 2i - 7 & \text{for } 1 \leq i < \frac{3n - 4z + 9}{2} \\
4z - 3n + 2i - 9 & \text{for } \frac{3n - 4z + 9}{2} \leq i \leq n.
\end{cases}
\]

Therefore, \( f^{'\prime}(V) \subseteq \{0, 2, 4, ..., 2s - 2\} \), where \( s = \text{max} \{p, q\} = n + 1 \). So, it follows that the vertex labels are all distinct and even. Hence, the graph

\( C_n(i), (n > 4), 3 \leq i \leq n - 1 \), cycle with one chord of odd order is \( k \)-even-edge-graceful for all \( k = z \left( \text{mod} \frac{q}{2} \right) \),

\( 0 \leq z \leq \frac{q}{2} - 1 \). \( \blacksquare \)

For example, consider the graph \( C_9(5) \). Here \( n = 9; s = \text{max} \{p, q\} = 10; 2s = 20 \).
The $15$-even-edge-graceful labeling of $C_9(5)$ is given in Figure 5.

For example, consider the graph $C_{11}(6)$. Here $n = 11; s = \max \{p, q\} = 12; 2s = 24$. The $11$-EEGL of $C_{11}(6)$ is given in Figure 6.

**Theorem 2.5**

The graph $C_n(i), (n \geq 4), 3 \leq i \leq n - 1$, cycle with one chord of even order is $k$-even-edge-graceful for all $k = z \mod q$, where $0 \leq z \leq q - 1$ and $n = 0 \mod 4$.

**Proof**

Let the vertices and edges be defined as in Theorem 6.3.2. First, we label the edges as follows:

For $k \geq 1$,

$$f\left( e_1 \right) = 2k + 1,$$

$$f\left( e_2 \right) = 2k - 1,$$

$$f\left( e_i \right) = 2k + 2i - 3 \quad \text{for } 3 \leq i < \frac{n}{2} + 3.$$  

For $k \geq 1$ and $\frac{n}{2} + 3 \leq i \leq n$,

$$f\left( e_i \right) = \begin{cases} 2k + 2i + 1 & \text{when } i \text{ is odd} \\ 2k + 2i - 3 & \text{when } i \text{ is even}. \end{cases}$$
\[ f(e_{n+1}) = \begin{cases} 
2k & \text{when } k = 0 \pmod{q} \\
2k + 2n - 2z + 2 & \text{when } k = z \pmod{q}, \ 1 \leq z \leq q - 1.
\end{cases} \]

Then the induced vertex labels are as follows:

**Case 1: \( k \equiv 0 \pmod{q} \)**

\[ f^*(v_1) = 2n - 2; \quad f^*(v_2) = 0. \]

\[ f^*(v_3) = 2. \]

\[ f^*(v_i) = \begin{cases} 
4i - 8 & \text{for } 4 \leq i < \frac{n}{2} + 3 \\
4i - 2n - 6 & \text{for } \frac{n}{2} + 3 \leq i \leq n.
\end{cases} \]

**Case 2: \( k \equiv z \pmod{q}, 1 \leq z \leq \frac{q - 1}{2} \)**

\[ f^*(v_1) = 4z + 4i - 8 \quad \text{when } i = 1, 2. \]

For \( k \equiv z \pmod{q}, 1 \leq z \leq \frac{q - 3}{2}, \)

\[ f^*(v_3) = 4z + 2 \]

For \( 4 \leq i \leq n, \)

\[ f^*(v_i) = \begin{cases} 
4z + 4i - 8 & \text{for } 4 \leq i < \frac{n}{2} - z + 3 \\
4z + 4i - 2n - 10 & \text{for } \frac{n}{2} - z + 3 \leq i < \frac{n}{2} + 3 \\
4z + 4i - 2n - 6 & \text{for } \frac{n}{2} + 3 \leq i < n - z + 2 \\
4z + 4i - 4n - 8 & \text{for } n - z + 2 \leq i \leq n.
\end{cases} \]

For \( k = \frac{q - 1}{2} \pmod{q} \),

\[ f^*(v_3) = 0 \]

\[ f^*(v_i) = \begin{cases} 
4i - 10 & \text{for } 4 \leq i < \frac{n}{2} + 3 \\
4i - 2n - 8 & \text{for } \frac{n}{2} + 3 \leq i \leq n.
\end{cases} \]

**Case 3: \( k = \frac{q + 1}{2} \pmod{q} \)**

\[ f^*(v_1) = 2n; \quad f^*(v_2) = 2. \]

\[ f^*(v_3) = 4. \]
\[ f^+(v_i) = \begin{cases} 4i - 6 & \text{for } 4 \leq i < \frac{n}{2} + 2 \\ 0 & \text{when } i = \frac{n}{2} + 2 \\ 4i - 2n - 4 & \text{for } \frac{n}{2} + 3 \leq i \leq n. \end{cases} \]

Case 4: \( k \equiv z \pmod{q}, \frac{q + 3}{2} \leq z \leq q - 1 \)

\[ f^+(v_i) = 4z + 4i - 2n - 10 \text{ when } i = 1, 2. \]
\[ f^+(v_3) = 4z - 2n. \]

For \( k \equiv (z \pmod{q}), \frac{q + 3}{2} \leq z \leq q - 2, \)

\[ f^+(v_i) = \begin{cases} 4z + 4i - 2n - 10 & \text{for } 4 \leq i < n - z + 3 \\ 4z + 4i - 4n - 12 & \text{for } n - z + 3 \leq i < \frac{n}{2} + 3 \\ 4z + 4i - 4n - 8 & \text{for } \frac{n}{2} + 3 \leq i < \frac{3n}{2} - z + 3 \\ 4z + 4i - 6n - 10 & \text{for } \frac{3n}{2} - z + 3 \leq i \leq n. \end{cases} \]

For \( k \equiv (q - 1) \pmod{q}, \)

\[ f^+(v_i) = \begin{cases} 4i - 12 & \text{for } 4 \leq i < \frac{n}{2} + 3 \\ 4i - 2n - 10 & \text{for } \frac{n}{2} + 3 \leq i \leq n. \end{cases} \]

Therefore, \( f^+(V) \subseteq \{0, 2, 4, ..., 2s - 2\}, \) where \( s = \max\{p, q\} = n + 1. \) So, it follows that the vertex labels are all distinct and even. Hence, the graph \( C_s(i), (n \geq 4), 3 \leq i \leq n - 1, \) cycle will one chord of even order is \( k \)-even-edge-graceful for all \( k \equiv z \pmod{q}, \) where \( 0 \leq z \leq q - 1 \) and \( n \equiv 0 \pmod{4}. \) 

For example, consider the graph \( C_{16}(5). \) Here \( n = 16; s = \max\{p, q\} = 17; 2s = 34. \)

The 18-EEGL of \( C_{16}(5) \) is given in Figure 7.

![Figure 7: 18-EEGL of C_{16}(5)](image-url)
Definition 2.6
The crown $C_n \oplus K_1$ is the graph obtained from the cycle $C_n$ by attaching pendant edge at each vertex of the cycle and is denoted by $C_n^+$. In this graph, $p = q = 2n$.

Theorem 2.7
The crown graph $C_n^+$ (if $n \geq 3$) of even order is $k$-even-edge-graceful for all $k = z \left( \frac{p}{3} \right)$, where $0 \leq z \leq \frac{p}{3} - 1$ and $n \equiv 0 \mod 3$.

Proof
For this graph, $p = q = 2n$. Let $v_1, v_2, \ldots, v_n$ and $v_1', v_2', \ldots, v_n'$ be the vertices and pendant vertices of $C_n^+$ respectively.

The edges are defined by

$e_i = (v_i, v_{i+1})$  \hspace{1cm} for $1 \leq i \leq n - 1$ ; \hspace{1cm} $e_n = (v_n, v_1)$

and $e_i' = (v_i', v_i')$ \hspace{1cm} for $1 \leq i \leq n$ (see Figure 8).

First, we label the edges as follows:
For $k \geq 1$,

$f(e_i) = 2k + 4i - 5$  \hspace{1cm} for $1 \leq i \leq n$  \hspace{1cm} 

$f(e_i') = 2k$  \hspace{1cm}  

$f(e_i) = 2k + 4(n - i + 1)$  \hspace{1cm} for $2 \leq i \leq n$.

Then the induced vertex labels are as follows:

Case 1: $k \equiv 0 \mod \frac{p}{3}$
\[ f^+(v_i) = \begin{cases} 4n + 4i - 10 & \text{when } i = 1, 2 \\ 4i - 10 & \text{for } 3 \leq i \leq n. \end{cases} \]

Case 2: \( k = z \left( \mod \frac{p}{3} \right), 1 \leq z \leq \frac{p}{3} - 1 \) and \( z \) is odd

\[ f^+(v_i) = \begin{cases} 6z + 4i - 10 & \text{for } 1 \leq i < n + \frac{5 - 3z}{2} \\ 6z - 4n + 4i - 10 & \text{for } n + \frac{5 - 3z}{2} \leq i \leq n. \end{cases} \]

Case 3: \( k = z \left( \mod \frac{p}{3} \right), 1 \leq z \leq \frac{p}{3} - 1 \) and \( z \) is even

\[ f^+(v_i) = \begin{cases} 6z + 4i - 10 & \text{for } 1 \leq i < n + \frac{6 - 3z}{2} \\ 6z - 4n + 4i - 10 & \text{for } n + \frac{6 - 3z}{2} \leq i \leq n. \end{cases} \]

The pendant vertices will have the labels \((\mod 2p)\) of the edges incident on them. Clearly, the vertex labels are all distinct and even. Hence, the crown graph \( C_n^+ \) \((n \geq 3)\) of even order is \(k\)-even-edge-graceful for all \( k = z \left( \mod \frac{p}{3} \right) \), where \( 0 \leq z \leq \frac{p}{3} - 1 \) and \( n = 0 \) (mod 3).

For example, consider the graph \( C_6^+ \). Here \( p = q = 12; s = \max\{p, q\} = 12; 2s = 24 \). The 2-even-edge-graceful labeling of \( C_6^+ \) is given in Figure 9.

![Figure 9: 2-EEGL of \( C_6^+ \)](image-url)
Theorem 2.8

The crown graph $C^*_n$, $(n \geq 4)$ of even order is $k$-even-edge-graceful for all $k = z \pmod{p}$, where $0 \leq z \leq p - 1$, $n \equiv 1 \pmod{3}$ and $n \neq 1$.

Proof

Let the vertices and edges be defined as in Theorem 6.4.2. The edge labels are also same as in Theorem 6.4.2.

Then the induced vertex labels are as follows:

Case 1: $k \equiv 0 \pmod{p}$

$$f^+(v_i) = \begin{cases} 4n + 4i - 10 & \text{when } i = 1, 2 \\ 4i - 10 & \text{for } 3 \leq i \leq n. \end{cases}$$

When $z$ is odd, the induced vertex labels are given below:

Case 2: $k \equiv z \pmod{p}$, $1 \leq z \leq \frac{p + 1}{3}$

$$f^+(v_i) = \begin{cases} 6z - 4n + 4i - 10 & \text{for } 1 \leq i < n + \frac{5 - 3z}{2} \\ 6z - 8n + 4i - 10 & \text{for } n + \frac{5 - 3z}{2} \leq i \leq n. \end{cases}$$

Case 3: $k \equiv z \pmod{p}$, $\frac{p + 4}{3} \leq z \leq \frac{2p + 2}{3}$

$$f^+(v_i) = \begin{cases} 6z - 4n + 4i - 10 & \text{for } 1 \leq i < 2n + \frac{5 - 3z}{2} \\ 6z - 8n + 4i - 10 & \text{for } 2n + \frac{5 - 3z}{2} \leq i \leq n. \end{cases}$$

Case 4: $k \equiv z \pmod{p}$, $\frac{2p + 5}{3} \leq z \leq p - 1$

$$f^+(v_i) = \begin{cases} 6z - 8n + 4i - 10 & \text{for } 1 \leq i < 3n + \frac{5 - 3z}{2} \\ 6z - 12n + 4i - 10 & \text{for } 3n + \frac{5 - 3z}{2} \leq i \leq n. \end{cases}$$

When $z$ is even, the induced vertex labels are given below:

Case 5: $k \equiv z \pmod{p}$, $1 \leq z \leq \frac{p + 1}{3}$

$$f^+(v_i) = \begin{cases} 6z + 4i - 10 & \text{for } 1 \leq i < n + \frac{6 - 3z}{2} \\ 6z - 4n + 4i - 10 & \text{for } n + \frac{6 - 3z}{2} \leq i \leq n. \end{cases}$$

Case 6: $k \equiv z \pmod{p}$, $\frac{p + 4}{3} \leq z \leq \frac{2p + 2}{3}$
The pendant vertices will have the labels \((\text{mod } 2^n)\) of the edges incident on them. Clearly, the vertex labels are all distinct and even. Hence, the crown graph \(C_n^+\), \((n \geq 4)\) of even order is \(k\)-even-edge-graceful for all \(k \equiv z \pmod{p}\), where \(0 \leq z \leq p - 1\), \(n \equiv 2 \pmod{3}\) and \(n \neq 2\). \(\blacksquare\)

For example, consider the graph \(C_7^+\). Here \(p = q = 14\); \(s = \max\{p, q\} = 14\); \(2s = 28\). The 5-even-edge-graceful labeling of \(C_7^+\) is given in Figure 10.

\[
f^+(v_i) = \begin{cases} 
6z - 4n + 4i - 10 & \text{for } 1 \leq i < 2n + \frac{6 - 3z}{2} \\
6z - 8n + 4i - 10 & \text{for } 2n + \frac{6 - 3z}{2} \leq i \leq n.
\end{cases}
\]

\textbf{Theorem 2.9}

The crown graph \(C_n^+\), \((n \geq 5)\) of even order is \(k\)-even-edge-graceful for all \(k \equiv z \pmod{p}\), where \(0 \leq z \leq p - 1\), \(n \equiv 2 \pmod{3}\) and \(n \neq 2\).

\textbf{Proof}

Let the vertices and edges be defined as in Theorem 2.7. The edge labels are also same as in Theorem 2.7. Then the induced vertex labels are as follows:

\textbf{Case 1: } k \equiv 0 \pmod{p}
When $z$ is odd, the induced vertex labels are given below:

Case 2: $k \equiv z \mod{p}$, $1 \leq z \leq \frac{p + 2}{3}$

$$f^+(v_i) = \begin{cases} 
6z + 4i - 10 & \text{for } 1 \leq i < n + \frac{5 - 3z}{2} \\
6z - 4n + 4i - 10 & \text{for } n + \frac{5 - 3z}{2} \leq i \leq n.
\end{cases}$$

Case 3: $k \equiv z \mod{p}$, $\frac{p + 5}{3} \leq z \leq \frac{2p + 1}{3}$

$$f^+(v_i) = \begin{cases} 
6z - 4n + 4i - 10 & \text{for } 1 \leq i < 2n + \frac{5 - 3z}{2} \\
6z - 8n + 4i - 10 & \text{for } 2n + \frac{5 - 3z}{2} \leq i \leq n.
\end{cases}$$

Case 4: $k \equiv z \mod{p}$, $\frac{2p + 4}{3} \leq z \leq p - 1$

$$f^+(v_i) = \begin{cases} 
6z - 8n + 4i - 10 & \text{for } 1 \leq i < 3n + \frac{5 - 3z}{2} \\
6z - 12n + 4i - 10 & \text{for } 3n + \frac{5 - 3z}{2} \leq i \leq n.
\end{cases}$$

When $z$ is even, the induced vertex labels are given below:

Case 5: $k \equiv z \mod{p}$, $1 \leq z \leq \frac{p + 2}{3}$

$$f^+(v_i) = \begin{cases} 
6z + 4i - 10 & \text{for } 1 \leq i < n + \frac{6 - 3z}{2} \\
6z - 4n + 4i - 10 & \text{for } n + \frac{6 - 3z}{2} \leq i \leq n.
\end{cases}$$

Case 6: $k \equiv z \mod{p}$, $\frac{p + 5}{3} \leq z \leq \frac{2p + 1}{3}$

$$f^+(v_i) = \begin{cases} 
6z - 4n + 4i - 10 & \text{for } 1 \leq i < 2n + \frac{6 - 3z}{2} \\
6z - 8n + 4i - 10 & \text{for } 2n + \frac{6 - 3z}{2} \leq i \leq n.
\end{cases}$$

Case 7: $k \equiv z \mod{p}$, $\frac{2p + 4}{3} \leq z \leq p - 1$

$$f^+(v_i) = \begin{cases} 
6z - 8n + 4i - 10 & \text{for } 1 \leq i < 3n + \frac{6 - 3z}{2} \\
6z - 12n + 4i - 10 & \text{for } 3n + \frac{6 - 3z}{2} \leq i \leq n.
\end{cases}$$
The pendant vertices will have the labels (mod 2p) of the edges incident on them. Clearly, the vertex labels are all distinct and even. Hence, the crown graph \( C_n^+ \), \((n \geq 5)\) of even order is \(k\)-even-edge-graceful for all \(k \equiv z \pmod{p}\), where \(0 \leq z \leq p - 1\), \(n = 2 \pmod{3}\) and \(n \neq 2\).

For example, consider the graph \( C_5^+ \). Here \(p = q = 10\); \(s = \max\{p, q\} = 10\); \(2s = 20\).

The 10-even-edge-graceful labeling of \( C_5^+ \) is given in Figure 11.

\[ \]

**Figure 11: 10-EEGL of \( C_5^+ \)**

**Definition 2.10**

A graph \( H_n(G) \) is obtained from a graph \( G \) by replacing each edge with \( m \) parallel edges.

**Theorem 2.11**

The graph \( H_5(P_n) \), \((n \geq 2)\) of even order is \(k\)-even-edge-graceful for all \(k \equiv z \pmod{p - 1}\), where \(0 \leq z \leq p - 2\) and \(n\) is even.

**Proof**

Let \(\{v_1, v_2, \ldots, v_n\}\) be the vertices of \( H_5(P_n) \). Let the edges \(e_i\) of \( H_5(P_n) \) be defined by \(e_i = (v_i, v_{i+1})\) for \(1 \leq i \leq n - 1\), \(1 \leq j \leq n\) (see Figure 12).

\[ \]

**Figure 12: \( H_5(P_n) \) with ordinary labeling**
For this graph, \( p = n; q = n(n - 1) \).

First, we label the edges as follows:

For \( k \geq 1 \),

\[
  f(e_{ij}) = \begin{cases} 
    2k + ni + 2j - 5 & \text{for } 1 \leq i \leq n - 1, \text{ } i \text{ is odd and } 1 \leq j \leq n \\
    2k + 2j + n(i - 1) - 4 & \text{for } 1 \leq i \leq n - 1, \text{ } i \text{ is even and } 1 \leq j \leq n.
  \end{cases}
\]

Then the induced vertex labels are as follows:

**Case 1:** \( k \equiv 0 \pmod{p - 1} \)

\[
  f^+(v_i) = 2n(n - 2).
\]

**Case 2:** \( k \equiv z \pmod{p - 1}, 1 \leq z \leq p - 2 \)

**Subcase (i):** \( k \equiv 1 \pmod{p - 1} \)

\[
  f^+(v_i) = n(2i - 3) \text{ for } 2 \leq i \leq n - 1.
\]

\[
  f^+(v_n) = n[n + 2(z - 2)].
\]

**Subcase (ii):** \( k \equiv z \pmod{p - 1}, 2 \leq z \leq \frac{p}{2} \)

\[
  f^+(v_i) = \begin{cases} 
    n \left( 2i - 3 \right) + 4n \left( z - 1 \right) & \text{for } 2 \leq i \leq n - 2z + 3 \\
    n \left[ 2 \left( i + 2z - n \right) - 5 \right] & \text{for } n - 2z + 3 \leq i \leq n - 1.
  \end{cases}
\]

\[
  f^+(v_n) = n[n + 2(z - 2)].
\]

**Subcase (iii):** \( k \equiv z \pmod{p - 1}, \frac{p + 2}{2} \leq z \leq p - 2 \)

\[
  f^+(v_i) = \begin{cases} 
    n \left[ 2 \left( i + 2z - n \right) - 5 \right] & \text{for } 2 \leq i < 2n - 2z + 2 \\
    n \left[ 2 \left( i - 2n + 2z \right) - 3 \right] & \text{for } 2n - 2z + 2 \leq i \leq n - 1.
  \end{cases}
\]

\[
  f^+(v_n) = n[2(z - 1) - n].
\]

Therefore, \( f^+(V) \subseteq \{0, 2, 4, ..., 2s - 2\} \), where \( s = \max \{p, q\} = n(n - 1) \). So, it follows that the vertex labels are all distinct and even. Hence, the graph \( H_s(P_n) \), \( n \geq 2 \) of even order is k-even-edge-graceful for all \( k \equiv z \pmod{p - 1} \), where \( 0 \leq z \leq p - 2 \) and \( n \) is even.

For example, consider the graph \( H_8(P_8) \). Here \( p = 8; q = 56 \);

\( s = \max \{p, q\} = 56; 2s = 112 \). The 6-EEGL of \( H_8(P_8) \) is given in Figure 13.
K-Even Edge-Graceful Labeling Of...

REFERENCES