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Entropy for Projection Operators Acting over a Probabilistic Space

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ABSTRACT : *In this work, we review some properties of the projection operators we have presented previously in other works for the purpose of obtain an expression for the entropy of operators acting on a probability space that is, considering the appropriate summation rules we developed in previous works. Then we establish the connection of this entropy with the matrix representation of those operators and discuss the role played by the dual space also with respect to entropy, a space we have previously defined, giving an academic example of how it works. Also we propose an expression of Entropy for a broad class of Hermitian operators which fulfill some requirements like be represented in matrix form, that permits to calculate the Entropy in some academic examples.*
KEYWORDS - *Entropy for Projection Operators, Group of Transformations, Hermitian Matrices, Symmetry Groups, Fredholm Equations, Electromagnetic Resonances.*

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I.INTRODUCTION

From Statistical Mechanics to Quantum Mechanics the concept of a space of probabilities has been taken as the association of the points in either a phase space or an ordinary one, and a function that represents the probabilistic density of the existence of a piece from a set constituted of an ensemble of systems which are compatible versions of a particular one in the sense of a behavior under specific limits of energy, momentum, angular momentum, etc. But we can take a different point of view that is we can think in a space constituted by the probabilities that a system is in a specific state, that is, each point is the probability that the system be in a specific state. For an example, we can think in a system that can be found in only two different states, say state 1 and state 2, and that the respective probabilities for each state are p and q, so p + q must sum 1. The space is the set of the two p and q probabilities. Suppose that p= o.3 and q=0.7. Then the projection of the space in the event p can be found by taking the complete space 1 and subtracting q=0.7, that is 0.3. Also the corresponding projection of the space in the event q can be found subtracting p=0.3 from the entire space 1 that is q=0.7. But what about the projection over the entire space $p+q$ over himself? Let us define the operators:

$$\Omega_p \equiv -q \text{ And} \tag{1}$$

And $$\Omega_q \equiv -p, \tag{2}$$

And because $\mathrm{p}+\mathrm{q}=1$ then

$$\Omega_p = p-1. \tag{3}$$

If we try to operate

$$(p+q)1 \tag{4}$$

We must build

$$\Omega_p + \Omega_q = (p-1)+(-p) = -1 \tag{5}$$

Which is obviously incorrect. Now, we can try in other way:

$$(\Omega_p+\Omega_q)1 = \Omega_p 1 + \Omega_q 1 = (1-q)+(1-p)$$
$$= [p+q-(1-p)]+[p+q-p] = 1 \tag{6}$$

That is: take 1 (the entire space) and subtract q so we have p; then we sum to this the result of now subtracting p to the entire space obtaining finally 1. So we must adjust the procedure to finally obtain the desired result 1. Before we propose even in a major dimension the convenient procedure we must note that the preceding two represents the fact that if we do not chose the appropriate rule then the conventional property of the complex and real numbers, vectors and matrices that is the associativity does not works for this kind of operators that we will call simply projection operators. Also we underline the possible application of the present work to the study of the Fredholm integral equations for which we have defined some integral projection operators in previous works [1].

II. The correct rule for adding projection operators

In this section we begin with a generalization of the rule for adding operators when we need to sum a higher number of them but first we will associate operators for the new probabilities and we denote the entire space as ξ that is:

$$\xi = \{p_i\} \qquad (7)$$

Where p_i is the probability to find the system in the i -th state and at the same time is a point of the set ξ and frequently we replace ξ for the complete probability to have the system in any state that is for the number 1:

$$\xi = 1 \qquad (8)$$

Let us denote:

$$\Omega_{p_i} \qquad (9)$$

For the corresponding projector operator over the state (or probability) p_i then, the sum of m projectors when the dimension of the space is n must be defined like:

$$(\sum_{i=1}^{m} \Omega_{p_i})\xi = \xi - \sum_{j \neq i}^{n} p_j = \xi - 0 = \xi \; (\sum_{i=1}^{m} \Omega_{p_i})\xi \equiv \xi - \sum_{j \neq i}^{n} p_j \qquad (10)$$

Because by the moment it is not necessary to show the case $\mathrm{m} \neq \mathrm{n}$ we take for example the case $\mathrm{m} = \mathrm{n} = 3$:

$$(\sum_{i=1}^{3} \Omega_{p_i})\xi = \xi - \sum_{j \neq i}^{3} p_j = \xi - 0 = \xi \quad \mathbf{(11)}$$

For m=2, n=3 that is for the sum over only two projectors is:

$$(\sum_{i=1}^{2} \Omega_{p_i})\xi = \xi - \sum_{j \neq i}^{3} p_j = p_1 + p_2 + p_3 - p_3 = p_1 + p_2 \qquad (12)$$

It is necessary to eliminate a restriction that appears in definition (10). If we try to sum:

$$p_3 + p_1 \qquad (13)$$

That is we want to sum from $\mathrm{i} = 3$ to $\mathrm{i} = 1$ or symbolically:

$$\sum_{\substack{i=3 \\ i \neq 2}}^{1} \Omega_{p_i} \equiv p_3 + p_1 \qquad (14)$$

This can be also written as a special case (with m=1, s=3) of:

$$\sum_{\substack{i=s \\ m<s}}^{m} \Omega_{p_i} \equiv \sum_{[i=1]}^{[m]} \Omega_{[p_i]} \qquad (15)$$

Where the parenthesis $[]$ signifies that we change the conventional order $i = 1, 2, \ldots, m, \ldots, n$ by a permutation $i = [1], [2], \ldots, [m], \ldots, [n]$ which in the case m=2, n=3 is: $i = [1], [2], [3]$ goes to $i = 3, 1, 2$. So in the general case equation (10) can be rewritten as:

$$(\sum_{[i=1]}^{[m]} \Omega_{[p_i]})\xi \equiv \xi - \sum_{[j \neq i]}^{[n]} p_{[j]} \qquad (16)$$

Now, we will obtain an expression for the Entropy for the projection operators acting over a probabilistic space by using a common tool in quantum mechanics, to this end, we define the following function:

$$\mathscr{H} = \sum_{[i=1]}^{[m]} [e^{\Omega_i}] \log[e^{\Omega_i}] \quad (17)$$

$$\mathscr{H} = \sum_{[i=1]}^{[m]} [e^{\Omega_i}][\Omega_i] \quad (18)$$

$$\mathscr{H} = \sum_{[i=1]}^{[m]} [e^{\Omega_i}][\Omega_i] \quad (19)$$

But we can develop the exponential and then:

$$\mathscr{H} = \sum_{[i=1]}^{[m]} \sum_{n=0}^{\infty} [\frac{\Omega_i}{n!}][\Omega_i] \quad (20)$$

$$\mathscr{H} = \sum_{[i=1]}^{[m]} \sum_{n=0}^{\infty} [\frac{\Omega_i^n}{n!}][\Omega_i] \quad (21)$$

Now acting on the entire space ξ :

$$\mathscr{H}\xi = \sum_{[i=1]}^{[m]} \sum_{n=0}^{\infty} [\frac{\Omega_i^{n+1}}{n!}]\xi \quad (22)$$

$$\mathscr{H}\xi = \sum_{[i=1]}^{[m]} \sum_{n=0}^{\infty} [\frac{p_i^{n+1}}{n!}] \quad (23)$$

$$\mathscr{H}\xi = \sum_{[i=1]}^{[m]} [e^{p_i}] \log[e^{p_i}] \quad (24)$$

Now we can make

$$\mathrm{X}_i = e^{p_i} \quad (25)$$

From this definition we can write:

$$\mathscr{H}\xi = \sum_{[i=1]}^{[m]} [\mathrm{X}_i] \log[\mathrm{X}_i] \quad (26)$$

From this expression we recognize that we can propose as a first attempt for the entropy of the projection operators $\{\Omega_i\}$:

$$H = -\log \sum_{[i=1]}^{[m]} [\mathrm{X}_i] \log[\mathrm{X}_i] = -\log \sum_{[i=1]}^{[m]} [e^{\Omega_i}] \log[e^{\Omega_i}]\xi \quad (27)$$

III. Matrix representation for the projector operators

The preceding rule expressed in equation (16) is useful for the purpose of give an understanding for the projection operators but we can go farther and find matrix representations that may also give an augmented vision of the possible implications to more complex systems than those former taken into account. In the process we will find additional properties like group properties and a dual space for this representations that must be interpreted but indeed have not known existence at this moment.

Let us associate to the space of probabilities for the case n=2 a vector notation:

$$\xi = \begin{pmatrix} p \\ q \end{pmatrix} \qquad (28)$$

Now, we define de four following projection operators:

$$\Omega_p = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad (29)$$

$$\Omega_q = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \qquad (30)$$

$$\Omega_\phi = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (31)$$

$$\Omega_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(Identity)} \qquad (32)$$

The multiplication table for the projectors (29-32)) is:

	Ω_I	Ω_p	Ω_q	Ω_ϕ
Ω_I	Ω_I	Ω_p	Ω_q	Ω_ϕ
Ω_p	Ω_p	$-i\Omega_\phi$	$i\Omega_I$	Ω_q
Ω_q	Ω_q	$i\Omega_I$	$i\Omega_\phi$	$-\Omega_p$
Ω_ϕ	Ω_ϕ	Ω_q	$-\Omega_p$	$-\Omega_I$

(33)

We can see that also the following rule is accomplished

$$\left(\Omega_{\mathrm{i}}\right)^4 = \Omega_{\mathrm{I}} \text{ With } i = I, p, q, \phi \qquad (34)$$

So the set of four matrices:

$$\chi = \left\{\Omega_p, \Omega_q, \Omega_I, \Omega_\phi\right\} \qquad (35)$$

Form an abelian cyclic group of fourth degree [2].

But we can verify that if we make the following associations:

$$p \Rightarrow \begin{pmatrix} p \\ iq \end{pmatrix} \equiv \left(1+i)\right) p \qquad (36)$$

$$q \Rightarrow \begin{pmatrix} ip \\ q \end{pmatrix} \equiv \left(1+i\right) q \qquad (37)$$

Then, $p, q,$ and ξ also they stick to the same rule (16):

$$(\Omega_p + \Omega_q)\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} - (0) = \begin{pmatrix} p \\ q \end{pmatrix} = \xi \qquad (38)$$

$$\Omega_p \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ iq \end{pmatrix} = \left(1+i\right) p \qquad (39)$$

$$\Omega_q \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ip \\ q \end{pmatrix} = (1+i)q \quad (40)$$

Also, we can verify that we can express the complete space ξ as the following sum:

$$p + q = \xi \qquad (41)$$

We can see that other possibilities exist:

$$\Omega_\phi \begin{pmatrix} p \\ q \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = i \begin{pmatrix} p \\ -q \end{pmatrix} \equiv \phi \quad (42)$$

And for example:

$$\Omega_\phi \phi = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} i \begin{pmatrix} p \\ -q \end{pmatrix} = -\begin{pmatrix} p \\ q \end{pmatrix} = -\xi \quad (43)$$

From equations (42) and (43) we can interpret ϕ as a dual space of probabilities and Ω_ϕ as the projection operator over that dual space.

Now, we calculate:

$$(\Omega_p + \Omega_q + \Omega_\varphi)\xi = \xi + \Omega_\varphi \xi = \xi + \phi \qquad (44)$$

It is a time to calculate the Entropy using definition (27) , but first we note that

$$\{\Omega_k\}\xi = p_k \quad (45)$$

Then we can see that a better definition must be taken (27) but with

$$e^{\Omega_k} \equiv e^{\{\Omega_k\}\xi} = e^{p_k} \quad (46)$$

That is:

$$H = -\log \sum_{[i=1]}^{[m]} [X_i] \log[X_i] = -\log \sum_{[i=1]}^{[m]} [e^{p_i}] \log[e^{p_i}] \quad (47)$$

So we can avoid the exponential forms and directly define for the Entropy of operators acting in a probabilistic space:

$$H = -\sum_{[i=1]}^{[m]} [\Omega_i] \log[\Omega_i] = -\sum_{[i=1]}^{[m]} [p_i] \log[p_i] \qquad (48)$$

Remember that the symbol $[\ ]$ represents a specific permutation between the desired $\{\Omega_i\}$

Let's evaluate H for the operator Ω_p for $p = 0.3$ and $q = 0.7$:

$$H = -(p)\log(p) \quad (49)$$

$$H = -0.3(\log(0.3)) = 0.521 \quad (50)$$

And for the operator Ω_q

$$H = -(q)\log(q) \quad (51)$$

$$H = -0.7(\log(0.7)) = 0.36 \quad (52)$$

For the sum $\Omega_p + \Omega_q$:

$$H = -0.3(\log(0.3)) - 0.7(\log(0.7) = 0.521 + 0.36 = 0.881 \quad (53)$$

We obtain the maximum Entropy for $p = 0.5$ and $q = 0.5$:

$$H = -0.5(\log(0.5)) - 0.5(\log(0.5) = 0.5 + 0.5 = 1 \quad (54)$$

But we can obtain the same result if we use new representations for the projection operators:

$$\Omega_\varphi = i\begin{pmatrix} p & 0 \\ 0 & -q \end{pmatrix} \quad (55)$$

$$\Omega_I = \frac{1}{2}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \quad (56)$$

$$\Omega_p = \begin{pmatrix} p & 0 \\ 0 & iq \end{pmatrix} \quad (57)$$

$$\Omega_q = \begin{pmatrix} ip & 0 \\ 0 & q \end{pmatrix} \quad (58)$$

And we can write a new version for Entropy appropriate for more general operators:

$$H = -\sum_{[i=1]}^{[m]}[\Omega_i]\log[\Omega_i] = -\sum_{[i=1]}^{[m]}[\mathrm{Re}\{Trace\{\Omega_i\}\}]\log[\mathrm{Re}\{Trace\{\Omega_i\}\}] \quad (59)$$

Let us apply eq. (59) first to operator (57):

$$H = -\Omega_p \log \Omega_p = -[\mathrm{Re}\left\{Trace\left\{\Omega_p\right\}\right\}]\log[\mathrm{Re}\left\{Trace\left\{\Omega_p\right\}\right\}] \quad (60)$$

$$H = -[\mathrm{Re}\left\{p + iq\right\}]\log[\mathrm{Re}\left\{p + iq\right\}] \quad (61)$$

$$H = -[p]\log[p] = -(.3)\log(.3) = 0.521 \quad (62)$$

That is the same as eq. (50)

Then we take operator (58) and apply definition (59):

$$H = -\Omega_q \log \Omega_q = -[\mathrm{Re}\left\{Trace\left\{\Omega_q\right\}\right\}]\log[\mathrm{Re}\left\{Trace\left\{\Omega_q\right\}\right\}] \quad (63)$$

$$H = -[\mathrm{Re}\left\{ip+q\right\}]\log[\mathrm{Re}\left\{ip+q\right\}] \quad (64)$$

$$H = -[q]\log[q] = -(.7)\log(.) = 0.36 \quad (65)$$

That is the same as eq. (52).
Up to this point, the expressions involved in the calculation of Entropy require that the trace of the matrices was different from zero, but if we wish to include Hermitian operators we must modify these expressions because the trace of a Hermitian matrix is zero. This problem can be saved thanks to the use of Pauli matrices.
Because Pauli matrices form a base for the description of every 2×2 Hermitian matrix, it is possible to include Hermitian operators whose trace is zero with the aid of them. The following are Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (66)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (67)$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (68)$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (69)$$

If $\{O_i\}$ is a set of Hermitian 2×2 operators we change slightly (59) to the form:

$$H = -\sum_{[l=0]}^{[s]}\sum_{[i=1]}^{[m]}[\lambda\sigma_l^{\dagger}O_i]\log[\lambda\sigma_l^{\dagger}O_i] = -\sum_{[l=0]}^{[s]}\sum_{[i=1]}^{[m]}[\mathrm{Re}\{Trace\{\lambda\sigma_l^{\dagger}O_i\}\}]\log[\mathrm{Re}\{Trace\{\lambda\sigma_l^{\dagger}O_i\}\}]$$
(70)

Provided that:

$$\sum_{[l=0]}^{[s]} \sum_{[i=1]}^{[m]} [\mathrm{Re}\{Trace\{\lambda\sigma_l^{\dagger}O_i\}\}] \leq 1 \qquad (71)$$

The symbol [ ] again represents permutation in the order of the indices (see equation (16)) and λ is a normalization factor.

We have for example wit $O_1 = \sigma_1$:

$$H = -\lambda\sigma_1^{\dagger}O_1 \log \lambda\sigma_1^{\dagger}O_1 = -\mathrm{Re}\{Trace\{\lambda\sigma_1^{\dagger}\sigma_1\} \log \mathrm{Re}\{Trace\{\lambda\sigma_l^{\dagger}\sigma_1\} \quad (72)$$

The Entropy will be:

$$H = -\mathrm{Re}\{2\lambda\} \log \mathrm{Re}\{2\lambda\} = -2\lambda \log 2\lambda = 0.2497 \quad (73)$$

With a normalization factor of:

$$\lambda = 0.0312 \qquad (74)$$

So we have for the four Pauli matrices:

$$H(\sigma_0) = 0.2497 \quad (75)$$

$$H(\sigma_1) = 0.2497 \quad (76)$$

$$H(\sigma_2) = 0.0.2497 \quad (77)$$

$$H(\sigma_3) = 0.2497 \quad (78)$$

And for the four matrices (the entire space of Pauli matrices):

$$H(\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3) = 0.9988 \quad (79)$$

That is, the Pauli matrices space behaves like a probabilistic space.

Now, if the Hermitian operator O_1 is:

$$O_1 = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} \quad (80)$$

$$= \alpha_0 I + \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3 \quad (81)$$

The Entropy will be:

$$H = -\sum_{l=0}^{3} [\mathrm{Re}\{Trace\{\lambda\sigma_l^{\dagger}O_1\}\}] \log[\mathrm{Re}\{Trace\{\lambda\sigma_l^{\dagger}O_1\}\}] \quad (82)$$

$$H = -[\mathrm{Re}\{Trace\{\lambda\sigma_0^{\dagger}O_1\}\}] \log[\mathrm{Re}\{Trace\{\lambda\sigma_0^{\dagger}O_1\}\}] \quad (83)$$

$$-[\mathrm{Re}\{Trace\{\lambda\sigma_1^{\dagger}O_1\}\}]\log[\mathrm{Re}\{Trace\{\lambda\sigma_1^{\dagger}O_1\}\}] \quad (84)$$

$$-[\mathrm{Re}\{Trace\{\lambda\sigma_2^{\dagger}O_1\}\}]\log[\mathrm{Re}\{Trace\{\lambda\sigma_2^{\dagger}O_1\}\}] \quad (85)$$

$$-[\mathrm{Re}\{Trace\{\lambda\sigma_3^{\dagger}O_1\}\}]\log[\mathrm{Re}\{Trace\{\lambda\sigma_3^{\dagger}O_1\}\}] \quad (86)$$

$$H = -2\lambda\alpha_0[\log[2\lambda\alpha_0] - 2\lambda\alpha_1[\log[2\lambda\alpha_1] - 2\lambda\alpha_2[\log[2\lambda\alpha_2] - 2\lambda\alpha_3[\log[2\lambda\alpha_3] \quad (87)$$

If we choose an arbitrary Hermitian matrix like:

$$O_1 = \begin{pmatrix} \alpha_0+\alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} = \begin{pmatrix} 2 & 1-i \\ 1+i & 0 \end{pmatrix} \quad (88)$$

Then $\alpha_i = 1$ and thus putting $\lambda = 0.0312$ we have:

$$H = -0.998977 \qquad (89)$$

We can think now in the operators $\{O_i\}$ like projectors acting on a probabilistic space as occurred in the case of abelian operators.

It is a very interesting fact that while the former projector operators defined over the probabilistic space ξ has been represented by means of members of abelian groups, the Hermitian operators $\{\sigma_i\}$ are the base of the $SU(2)$ (non-abelian) group.

Nevertheless, the expression for the Entropy it´s very similar.

IV.CONCLUSION

On this paper we have giving explicit expressions for the Entropy in the cases of projection operators defined over a space of probabilities that shows a hidden symmetry when are represented in abelian matrix form but also we give an alternative Entropy for operators represented by Hermitian operators, as we know, Hermitian matrices are traceless so we must modify slightly our former expressions to include these important class of operators. Then we have taken our proposed expressions and calculate explicitly Entropy values for distinct situations and compare. We observe that it is possible to normalize the probabilistic space and compare very different projectors thanks to the use of appropriate expression for the Entropy as we have recognize from eqs. (62), (65) and (89) that represents Entropy values for two different types of operators: abelian and Hermitian. We consider that the present work can be taken as a first important step for analyzing a very broad class of subjects from communications, probability spaces to qubits.

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