Fuzzy strongly c -semi irresolute and fuzzy strongly c -semi continuous functions in fuzzy topological spaces

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ABSTRACT: In this paper we introduce the concept of fuzzy strongly C – semi interior and fuzzy strongly C – semi closure operators by using arbitrary complement function C of a fuzzy topological space where C : [0, 1] → [0, 1] is a function and investigate some of their basic properties of a fuzzy topological space.

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I. INTRODUCTION

The concept of complement function is used to define a fuzzy closed subset of a fuzzy topological space. That is a fuzzy subset λ is fuzzy closed if the standard complement 1−λ = λ′ is fuzzy open. Here the standard complement is obtained by using the function C : [0, 1] → [0, 1] defined by C (x) = 1−x, for all x ∈ [0, 1]. Several fuzzy topologists used this type of complement while extending the concepts in general topological spaces to fuzzy topological spaces. But there are other complements in the fuzzy literature [10]. This motivated the author to introduce the concepts of fuzzy C -closed sets and fuzzy C - semi closed sets in fuzzy topological spaces, where C : [0, 1] → [0, 1] is an arbitrary complement function.

Bin [6] defined the notion of fuzzy strongly semi interior and fuzzy strongly semi closure operators in fuzzy topological spaces and studied their properties. In this paper, we generalize the concept of fuzzy strongly semi interior and fuzzy strongly semi closure operators by using the arbitrary complement function C, instead of the usual fuzzy complement function, fuzzy strongly C – semi interior instead of fuzzy strongly semi interior and by using fuzzy strongly C – semi closure instead of fuzzy strongly semi closure.

For the basic concepts and notations, one can refer Chang [7]. The concepts that are needed in this paper are discussed in the second section. The concepts of fuzzy strongly C – semi interior and strongly C – semi closure operators in fuzzy topological spaces and studied their properties in the third section. The fourth section is devoted to the applications of fuzzy strongly C – semi open and fuzzy strongly C – semi closed sets to fuzzy continuous functions and fuzzy irresolute functions.

II. PRELIMINARIES

Throughout this paper (X, τ) denotes a fuzzy topological space in the sense of Chang. Let C : [0, 1] → [0, 1] be a complement function. If λ is a fuzzy subset of (X, τ) then the complement C λ of a fuzzy subset λ is defined by C λ(x) = C (λ(x)) for all x ∈ X. A complement function C is said to satisfy

(i) the boundary condition if C (0) = 1 and C (1) = 0,
(ii) monotonic condition if x ≤ y ⇒ C (x) ≥ C (y), for all x, y ∈ [0, 1],
(iii) involutive condition if C (C (x)) = x, for all x ∈ [0, 1].

The properties of fuzzy complement function C and C λ are given in George Klir [8] and Bageerathi et al [2]. The following lemma will be useful in sequel.

Definition 2.1 [Definition 3.1, [2]]

Let (X, τ) be a fuzzy topological space and C be a complement function. Then a fuzzy subset λ of X is fuzzy C -closed in (X, τ) if C λ is fuzzy open in (X, τ).

Lemma 2.2 [Proposition 3.2, [2]]

Let (X, τ) be a fuzzy topological space and C be a complement function that satisfies the involutive condition. Then a fuzzy subset λ of X is fuzzy open in (X, τ) if C λ is a fuzzy C -closed subset of (X, τ).
**Definition 2.3** [Definition 4.1, [2]]

Let \((X, \tau)\) be a fuzzy topological space. Then for a fuzzy subset \(\lambda\) of \(X\), the fuzzy \(\mathcal{C}\) -closure of \(\lambda\) is defined as the intersection of all fuzzy \(\mathcal{C}\) -closed sets \(\mu\) containing \(\lambda\). The fuzzy \(\mathcal{C}\) -closure of \(\lambda\) is denoted by \(\text{Cl}_\mathcal{C} \lambda\), that is equal to \(\bigwedge \{\mu : \mu \supseteq \lambda, \mathcal{C}\mu \supseteq \tau\}\).

**Lemma 2.4** [Lemma 4.2, [2]]

If the complement function \(\mathcal{C}\) satisfies the monotonic and involutive conditions, then for any fuzzy subset \(\lambda\) of \(X\), (i) \(\mathcal{C}(\text{Int} \lambda) = \text{Cl}_\mathcal{C} \mathcal{C}(\lambda)\) and (ii) \(\mathcal{C}(\text{Cl}_\mathcal{C} \lambda) = \text{Int} \mathcal{C}(\lambda)\).

**Definition 2.5** [Definition 2.15, 3]

A fuzzy topological space \((X, \tau)\) is \(\mathcal{C}\) -product related to another fuzzy topological space \((Y, \sigma)\) if for any fuzzy subset \(\nu\) of \(X\) and \(\zeta\) of \(Y\), whenever \(\mathcal{C}\lambda \supseteq \nu\) and \(\mathcal{C}\mu \supseteq \zeta\) imply \(\mathcal{C}\lambda \times 1 \cap 1 \times \mathcal{C}\mu \supseteq \nu \times \zeta\), where \(\lambda \in \tau\) and \(\mu \in \sigma\), there exist \(\lambda_1 \in \tau\) and \(\mu_1 \in \sigma\) such that \(\mathcal{C}\lambda_1 \supseteq \nu\) or \(\mathcal{C}\mu_1 \supseteq \zeta\) and \(\mathcal{C}\lambda_1 \times 1 \cap 1 \times \mathcal{C}\mu_1 = \mathcal{C}\lambda \times 1 \cap 1 \times \mathcal{C}\mu\).

**Definition 2.6** [Definition 3.1, [7 & 8]]

A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is called fuzzy \(\mathcal{C}\) -semi continuous if \(f^{-1}(\mu)\) is a fuzzy \(\mathcal{C}\) -semi open set of \(X\) for each fuzzy open set \(\mu\) of \(Y\).

**Lemma 2.7** [1]

Let \(\{\lambda_\alpha\}\) be the family of fuzzy subsets of a fuzzy topological space \(X\). Then \(\bigvee \text{Cl}_\mathcal{C} (\lambda_\alpha) \leq \text{Cl}_\mathcal{C} (\bigvee \lambda_\alpha)\).

**Lemma 2.8** [Lemma 5.1, [2]]

Suppose \(f\) is a function from \(X\) to \(Y\). Then \(f^{-1}(\mathcal{C}\mu ) = \mathcal{C}(f^{-1}(\mu))\) for any fuzzy subset \(\mu\) of \(Y\).

**III. FUZZY STRONGLY \(\mathcal{C}\) – SEMI INTERIOR AND FUZZY STRONGLY \(\mathcal{C}\) – SEMI CLOSURE**

In this section, we define the concept of fuzzy strongly \(\mathcal{C}\) – semi interior and fuzzy strongly \(\mathcal{C}\) – semi closure and investigate some of their basic properties.

**Definition 3.1**

Let \((X, \tau)\) be a fuzzy topological space and \(\mathcal{C}\) be a complement function. Then for a fuzzy subset \(\lambda\) of \(X\), the fuzzy strongly \(\mathcal{C}\) – semi interior of \(\lambda\) (briefly \(\text{SISI}_\mathcal{C} \lambda\)), is the union of all fuzzy strongly \(\mathcal{C}\) – semi open sets of \(X\) contained in \(\lambda\).

That is, \(\text{SISI}_\mathcal{C} \lambda = \bigvee \{\mu : \mu \subseteq \lambda, \mu \text{ is fuzzy strongly }\mathcal{C} \text{ – semi open}\}\).

**Proposition 3.2**

Let \((X, \tau)\) be a fuzzy topological space and let \(\mathcal{C}\) be a complement function that satisfies the monotonic and involutive conditions. Then for any fuzzy subsets \(\lambda\) and \(\mu\) of a fuzzy topological space \(X\), we have

(i) \(\text{SISI}_\mathcal{C} \lambda \subseteq \lambda\),

(ii) \(\lambda\) is fuzzy strongly \(\mathcal{C}\) – semi open \(\iff\) \(\text{SISI}_\mathcal{C} \lambda = \lambda\),

(iii) \(\text{SISI}_\mathcal{C} (\text{SISI}_\mathcal{C} \lambda) = \text{SISI}_\mathcal{C} \lambda\),

(iv) If \(\lambda \subseteq \mu\), then \(\text{SISI}_\mathcal{C} \lambda \subseteq \text{SISI}_\mathcal{C} \mu\).

**Proof.**

(i) follows from Definition 3.1.

Let \(\lambda\) be fuzzy strongly \(\mathcal{C}\) – semi open. Since \(\lambda \subseteq \lambda\), by using Definition 3.1, \(\lambda \subseteq \text{SISI}_\mathcal{C} \lambda\). By using (i), we get \(\text{SISI}_\mathcal{C} \lambda = \lambda\). Conversely, we assume that \(\text{SISI}_\mathcal{C} \lambda = \lambda\). By using Definition 3.1, \(\lambda\) is fuzzy strongly \(\mathcal{C}\) – semi open. Thus (ii) is proved.

By using (ii), we get \(\text{SISI}_\mathcal{C} (\text{SISI}_\mathcal{C} \lambda) = \text{SISI}_\mathcal{C} \lambda\). This proves (iii).

Since \(\lambda \subseteq \mu\), by using (i), \(\text{SISI}_\mathcal{C} \lambda \subseteq \lambda \subseteq \mu\). By using Definition 3.1, \(\mu \subseteq \text{SISI}_\mathcal{C} \mu\). This implies that \(\text{SISI}_\mathcal{C} \lambda \subseteq \text{SISI}_\mathcal{C} \mu\). This proves (iv).

**Proposition 3.3**

Let \((X, \tau)\) be a fuzzy topological space and let \(\mathcal{C}\) be a complement function that satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets \(\lambda\) and \(\mu\) of a fuzzy topological space, we have

(i) \(\text{SISI}_\mathcal{C} (\lambda \cup \mu) \supseteq \text{SISI}_\mathcal{C} \lambda \cup \text{SISI}_\mathcal{C} \mu\)

and (ii) \(\text{SISI}_\mathcal{C} (\lambda \cap \mu) \supseteq \text{SISI}_\mathcal{C} \lambda \cap \text{SISI}_\mathcal{C} \mu\).
Proof.

Since \( \lambda \leq \lambda \lor \mu \) and \( \mu \leq \lambda \lor \mu \), by using Proposition 3.2(iv), we get \( \text{SSInt}_C \lambda \leq \text{SSInt}_C (\lambda \lor \mu) \) and \( \text{SSInt}_C \mu \leq \text{SSInt}_C (\lambda \lor \mu) \). This implies that \( \text{SSInt}_C \lambda \lor \text{SSInt}_C \mu \leq \text{SSInt}_C (\lambda \lor \mu) \).

Since \( \lambda \land \mu \leq \lambda \) and \( \land \mu \leq \mu \), by using Proposition 3.2(iv), we get \( \text{SSInt}_C (\lambda \land \mu) \leq \text{SSInt}_C \lambda \) and \( \text{SSInt}_C (\lambda \land \mu) \leq \text{SSInt}_C \mu \). This implies that \( \text{SSInt}_C (\lambda \land \mu) \leq \text{SSInt}_C \lambda \land \text{SSInt}_C \mu \).

**Definition 3.4**

Let \((X, \tau)\) be a fuzzy topological space. Then for a fuzzy subset \( \lambda \) of \( X \), the fuzzy strongly \( C \) – semi closure of \( \lambda \) (briefly \( \text{SSCl}_C \lambda \)), is the intersection of all fuzzy strongly \( C \) – semi closed sets containing \( \lambda \). That is \( \text{SSCl}_C \lambda = \land \{ \mu : \mu \geq \lambda, \mu \) is fuzzy strongly \( C \) – semi closed \}.

The concepts of “fuzzy strongly \( C \) – semi closure” and “fuzzy strongly semi closure” are identical if \( C \) is the standard complement function.

**Proposition 3.5**

If the complement functions \( C \) satisfies the monotonic and involutive conditions. Then for any fuzzy subset \( \lambda \) of \( X \) (i) \( \text{SSInt}_C \lambda \leq \text{SSCl}_C \lambda \provider (\lambda \land \lambda) \), and (ii) \( \text{SSCl}_C \lambda = \text{SSInt}_C \lambda \), where \( \text{SSInt}_C \lambda \) is the union of all fuzzy strongly \( C \) – semi open sets contained in \( \lambda \).

**Proof.**

By using Definition 3.1, \( \text{SSInt}_C \lambda = \land \{ \mu : \mu \leq \lambda, \mu \) is fuzzy strongly \( C \) – semi open \}. Taking complement on both sides, we get \( C (\text{SSInt}_C \lambda) = \text{Cl} \lambda \land \text{Cl} \lambda \). This implies that \( C (\text{SSInt}_C \lambda) = \text{Cl} \lambda \land \text{Cl} \lambda \). Then for the fuzzy subsets \( \lambda \) and \( \mu \) of a fuzzy topological space \( X \), we have

(i) \( \lambda \leq \text{SSCl}_C \lambda \),

(ii) \( \lambda \) is fuzzy strongly \( C \) – semi closed \( \Leftrightarrow \text{SSCl}_C \lambda = \lambda \),

(iii) \( \text{SSCl}_C \text{SSCl}_C \lambda = \text{Cl} \lambda \),

(iv) \( \lambda \leq \mu \) then \( \text{SSCl}_C \lambda \leq \text{SSCl}_C \mu \).

**Proof.**

The proof for (i) follows from \( \text{SSCl}_C \lambda = \text{Cl} \lambda \land \mu \), \( \mu \) is fuzzy strongly \( C \) – semi closed. Since \( \text{SSInt}_C \lambda \) satisfies the monotonic and involutive conditions. Then by using Proposition 5.3 in [5], \( C \) \( \lambda \) is fuzzy strongly \( C \) – semi open. By using Proposition 3.2(ii) in [5], \( \text{SSInt}_C \lambda = \text{Cl} \lambda \). By using Proposition 3.5, \( \text{SSCl}_C \lambda = \text{Cl} \lambda \). Taking complement on both sides, we get \( \text{Cl} \lambda = \text{Cl} \lambda \). Since the complement function satisfies the involutive condition, \( \text{SSCl}_C \lambda = \lambda \).

Conversely, we assume that \( \text{SSCl}_C \lambda = \lambda \). Taking complement on both sides, we get \( \text{Cl} \lambda = \text{Cl} \lambda \). By using Proposition 3.5, \( \text{SSInt}_C \lambda = \text{Cl} \lambda \). By using Proposition 3.2(ii) in [5], \( C \) \( \lambda \) is fuzzy strongly \( C \) – semi open. Again by using Proposition 5.3 in [5], \( \lambda \) is fuzzy strongly \( C \) – semi closed. Thus (ii) proved.

By using Proposition 3.5, \( \text{SSCl}_C \lambda = \text{SSInt}_C \lambda \). This implies that \( \text{Cl} \lambda \) is fuzzy strongly \( C \) – semi open. By using Proposition 5.3 in [5], \( \text{SSCl}_C \lambda \) is fuzzy strongly \( C \) – semi closed. By applying (ii), we have \( \text{SSInt}_C \text{SSCl}_C \lambda = \text{SSCl}_C \lambda \). This proves (iii).

Suppose \( \lambda \leq \mu \). Since \( \text{Cl} \lambda \) satisfies the monotonic condition, \( \text{Cl} \lambda \) is fuzzy strongly \( C \) – semi open. By using Proposition 5.3 in [5], \( \text{SSCl}_C \lambda \) is fuzzy strongly \( C \) – semi closed. By applying (ii), we have \( \text{SSInt}_C \text{SSCl}_C \lambda = \text{SSCl}_C \lambda \). This proves (iv).

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Proposition 3.7
Let \((X, \tau)\) be a fuzzy topological space and \(C\) be a complement function that satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets \(\lambda\) and \(\mu\) of a fuzzy topological space, we have (i) \(SSC\lambda (\lambda \land \mu) = SSC\lambda \land SSC\lambda \mu\) and (ii) \(SSC\lambda (\lambda \land \mu) \leq SSC\lambda \land SSC\lambda \mu\).

**Proof.**
Since \(C\) satisfies the involutive condition, \(SSC\lambda (\lambda \land \mu) = SSC\lambda (C (\lambda \land \mu)).\) Since \(C\) satisfies the monotonic and involutive conditions, by using Proposition 3.5, \(SSC\lambda (\lambda \land \mu) = SSC\lambda (\lambda \land \mu).\) Using Lemma 2.10 in [2], \(SSC\lambda (\lambda \land \mu) = C (SSC\lambda (C (\lambda \land \mu)).\) Again by Proposition 3.3, \(SSC\lambda (\lambda \land \mu) \leq C (SSC\lambda (\lambda \land \mu)).\) By using Proposition 3.5, \(SSC\lambda (\lambda \land \mu) \leq SSC\lambda (\lambda \land \mu).\) Also \(SSC\lambda (\lambda \land \mu) \leq SSC\lambda (\lambda \land \mu)\) and \(SSC\lambda (\mu) \leq SSC\lambda (\lambda \land \mu)\) that implies \(SSC\lambda (\lambda \land \mu) \leq SSC\lambda (\lambda \land \mu)\). It follows that \(SSC\lambda (\lambda \land \mu) = SSC\lambda \land SSC\lambda \mu.\)

Since \(SSC\lambda (\lambda \land \mu) \leq SSC\lambda \land \mu\) and \(SSC\lambda (\lambda \land \mu) \leq SSC\lambda \land \mu\), it follows that \(SSC\lambda (\lambda \land \mu) \leq SSC\lambda \land \mu.\)

Proposition 3.8
Let \((X, \tau)\) be a fuzzy topological space and \(C\) be a complement function that satisfies the monotonic and involutive conditions. Then for any family \(\{\lambda_\alpha\}\) of fuzzy subsets of a fuzzy topological space, we have (i) \(\lor (SSC\lambda \lambda_\alpha) \leq SSC\lambda (\lor \lambda_\alpha)\) and (ii) \(\land (SSC\lambda \lambda_\alpha) \leq \land (SSC\lambda \lambda_\alpha)\).

**Proof.**
For every \(\beta, \lambda \land \mu \leq SSC\lambda (\lor \lambda_\alpha)\). By using Proposition 3.6(iv), \(SSC\lambda \lambda \land \mu \leq SSC\lambda (\lor \lambda_\alpha)\) for every \(\beta\). This implies that \(SSC\lambda \lambda_\alpha \leq SSC\lambda (\lor \lambda_\alpha)\). This proves (i). Now \(\land \lambda_\alpha \leq \lambda \land \mu\) for every \(\beta\). Again using Proposition 3.6(iv), we get \(SSC\lambda (\land \lambda_\alpha) \leq SSC\lambda \lambda_\alpha\). This implies that \(SSC\lambda (\land \lambda_\alpha) \leq \land (SSC\lambda \lambda_\alpha)\). This proves (ii).

Proposition 3.9
Let \((X, \tau)\) be a fuzzy topological space and \(C\) be a complement function that satisfies monotonic and involutive properties. Let \(\lambda\) be a fuzzy subset of a fuzzy topological space \(X\). Then (i) \(Cl\lambda (SSC\lambda \lambda) = Cl\lambda\) and (ii) \(Cl\lambda \land Cl\lambda \land = SSC\lambda \lambda\).

**Proof.**
By using Proposition 3.6, we have \(\lambda \leq SSC\lambda \lambda \leq Cl\lambda \land Cl\lambda \land\). Since \(C\) satisfies the monotonic and involutive conditions, by using Lemma 2.6 in [2], \(Cl\lambda \land Cl\lambda \land = Cl\lambda \land Cl\lambda \land\). Then \(Cl\lambda (SSC\lambda \lambda) = Cl\lambda \land Cl\lambda \land\). This proves (i).

Since \(SSC\lambda \lambda\) is fuzzy strongly \(C\) – semi closed, by using Proposition 5.2 in [5], \(Cl\lambda \land Cl\lambda \land \lambda \leq SSC\lambda \lambda\). By using (i), \(Cl\lambda (SSC\lambda \lambda) = Cl\lambda \land Cl\lambda \land\). This implies that \(Cl\lambda \land Cl\lambda \land \leq SSC\lambda \lambda\).

**IV. APPLICATIONS**
This section is devoted to the application of fuzzy strongly \(C\) – semi open and fuzzy strongly \(C\) – semi closed sets to fuzzy continuous and fuzzy irresolute functions.

**Definition 4.1**
Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a function. Then \(f\) is said to be (i) fuzzy strongly \(C\) – semi irresolute if \(f^{-1}(\delta)\) is fuzzy strongly \(C\) – semi open in \(X\) for each fuzzy strongly \(C\) – semi open set \(\delta\) in \(Y\).

(ii) fuzzy strongly \(C\) – semi continuous if \(f^{-1}(\delta)\) is fuzzy strongly \(C\) – semi open in \(X\) for each fuzzy open set \(\delta\) in \(Y\).

**Remark 4.2**
It is clear that every fuzzy strongly \(C\) – semi irresolute mapping is fuzzy strongly \(C\) – semi continuous and every fuzzy strongly \(C\) – semi continuous mapping is fuzzy \(C\) – semi continuous. But the converses need not be as shown by the following example.

**Example 4.3**
Consider the identity map \(f: (X, \tau) \rightarrow (Y, \sigma)\). Let \(X = Y = \{a, b, c\}\), \(\tau = \{0, 1, \lambda, \mu, \lambda \land \mu, \lambda \lor \mu\}\), where \(\lambda = \{a_0, b_0, c_0\}\) and \(\mu = \{a_0, b_0, c_0\}\). Now \(\sigma = \{0, 1, \eta\}\), where \(\eta = \{a_0, b_0, c_0\}\). Let \(C(x) = \frac{2x}{1 + x}, 0 \leq x \leq 1\), be the complement function. Here \(f^{-1}(\eta)\) is fuzzy \(C\) – semi open but not fuzzy strongly \(C\) – semi open. Hence \(f\) is fuzzy \(C\) – semi continuous but not fuzzy strongly \(C\) – semi continuous.
Example 4.4
Consider the identity map \( f : (X, \tau) \to (Y, \sigma) \), where \( X = Y = \{ a, b, c \}, \tau = \{ \emptyset, \{ a, b \}, \{ a \} \}, \{ a, b, c \} \), \( \sigma = \{ \emptyset, \{ a \}, \{ b \} \} \). The family of all fuzzy \( C \)-closed sets \( \mathcal{C} \) satisfies the following condition.

Theorem 4.5
Let \( f : (X, \tau) \to (Y, \sigma) \) and \( C \) be a complement function that satisfies the involutive condition. Then the following conditions are equivalent.

(a) \( f \) is fuzzy strongly \( C \) – semi irresolute.
(b) \( f^1(\gamma) \) is fuzzy strongly \( C \) – semi closed in \( X \) for each fuzzy \( C \) – semi closed set \( \gamma \) in \( Y \).
(c) For each fuzzy set \( \lambda \) in \( X \), \( f(\text{SSCl}_C(\lambda)) \leq \text{SSCl}_C(f(\lambda)) \).
(d) For each fuzzy set \( \mu \) in \( Y \), \( \text{SSCl}_C(f^1(\mu)) \leq f^1(\text{SSCl}_C(\mu)) \).

Proof
(a) \( \Rightarrow \) (b): Let \( \gamma \) be a fuzzy strongly \( C \) – semi closed in \( Y \). Since the complement function \( C \) satisfies the monotonic and involutive conditions, by using Proposition 5.3 in [5], \( C \gamma \) is fuzzy strongly \( C \) – semi open in \( Y \). By using Definition 4.1, \( f^1(C \gamma) \) is fuzzy strongly \( C \) – semi open in \( Y \) which implies that \( C(f^1(\gamma)) \) is fuzzy strongly \( C \) – semi open in \( X \). Again by using Proposition 5.3 in [5], \( f^1(\gamma) \) is fuzzy strongly \( C \) – semi closed in \( X \).

(b) \( \Rightarrow \) (c): Let \( \lambda \) be a fuzzy set in \( X \). Then \( \text{SSCl}_C(f(\lambda)) \) is fuzzy strongly \( C \) – semi closed in \( Y \). Let \( \lambda \) be a fuzzy set in \( X \). By using (ii), \( f^1(\text{SSCl}_C(\lambda)) \) is fuzzy strongly \( C \) – semi closed in \( X \). Since the complement function \( C \) satisfies the monotonic and involutive conditions, by using Proposition 3.5, \( \text{SSCl}_C(\lambda) \leq \text{SSCl}_C(f^1(\text{SSCl}_C(\lambda))) = f^1(\text{SSCl}_C(f(\lambda))) \). This implies that \( f(\text{SSCl}_C(\lambda)) \leq f^1(\text{SSCl}_C(f(\lambda))) \).

(c) \( \Rightarrow \) (d): Let \( \mu \) be a fuzzy set in \( Y \). By using (c), \( f(\text{SSCl}_C(f^1(\mu))) \leq \text{SSCl}_C(f(f^1(\mu))) \leq \text{SSCl}_C(\mu) \). This implies \( f^1(\text{SSCl}_C(f^1(\mu))) \leq f^1(\text{SSCl}_C(\mu)) \).

(d) \( \Rightarrow \) (e): Let \( \gamma \) be a fuzzy strongly \( C \) – semi closed in \( X \). Then by using our assumption, \( \text{SSCl}_C(f^1(\gamma)) \) is fuzzy strongly \( C \) – semi closed in \( Y \). By using Definition 4.1, \( f^1(\gamma) \) is fuzzy strongly \( C \) – semi open in \( X \).

Theorem 4.6
Let \( f : X \to Y \) be a mapping. Let \( C \) be a complement function that satisfies monotonic and involutive conditions. Then the following are equivalent.

(a) \( f \) is fuzzy strongly \( C \) – semi continuous
(b) \( f^1(\gamma) \) is fuzzy strongly \( C \) – semi closed set in \( X \) for each fuzzy \( C \) – closed set \( \gamma \) in \( Y \).
(c) \( f(\text{SSCl}_C(\lambda)) \leq \text{SSCl}_C(f(\lambda)) \) for each fuzzy set \( \lambda \) in \( X \).
(d) \( \text{SSCl}_C(f^1(\mu)) \leq f^1(\text{SSCl}_C(\mu)) \) for each fuzzy set \( \mu \) in \( Y \).

Proof
Let \( \gamma \) be a fuzzy \( C \) -closed set in \( Y \). By using Definition 2.1, \( C \gamma \) is fuzzy open. By using Definition 4.1, \( f^1(C \gamma) \) is fuzzy strongly \( C \) – semi open. Since \( C(f^1(\gamma)) = C(f^1(\gamma)) \), that implies \( C \) is fuzzy strongly \( C \) – semi open. Since the complement function \( C \) satisfies the monotonic and involutive conditions, by using Proposition 5.3 in [5], \( f^1(\gamma) \) is fuzzy strongly \( C \) – semi closed in \( X \).

Conversely, let \( \delta \) be a fuzzy open set in \( X \). By using Lemma 2.7 in [2], \( f^1(C \delta) = C(f^1(\delta)) \) is fuzzy strongly \( C \) – semi closed in \( X \). Therefore \( f^1(\gamma) \) is fuzzy strongly \( C \) – semi closed in \( X \).
REFERENCES