International Journal of Engineering Science Invention
ISSN (Online): 2319 – 6734, ISSN (Print): 2319 – 6726
www.ijesi.org ||Volume 4 Issue 11|| November 2015 || PP.13-17

Inversion Theorem for Generalized Fractional Hilbert Transform

Akilahmad Sheikh[1], Alka Gudadhe[2]

[1]*Department of applied mathematics, Tulsiramji Gaikwad-Patil College of Engineering and Technology, Nagpur, India*

[2]*Department of mathematics, Government Vidarbha Institute of Science and Humanities, Amravati, India*

Abstract: *The generalized fractional Hilbert transform has been used in several areas, including optical, signal processing. In this paper, we have proved inversion theorem for generalized fractional Hilbert transform using Zemanian technique.*
Key Words: *Hilbert transform, generalized fractional Hilbert Transform, Signal Processing.*

I. INTRODUCTION

Fractional integral transforms play an important role in signal processing, image reconstruction, pattern recognition [7, 8]. The Fourier transform plays a major role in the theory of optics, signal processing and many other branch of engineering. In 1980 Victor Namias [5] introduced generalization of Fourier transform, the fractional Fourier Transform (FrFT) through eigen value method which he applied it to quantum mechanics. Zayed [9] has discussed about the product and convolution concerning that transform. Numbers of other integral transforms also have been extended in the fractional domain for example, Akay [1] had studied fractional Mellin transform, Gudadhe, Joshi [3] discussed generalized Half Linear Canonical transform, Sontakke, Gudadhe [6] studied number of operation transform formulae of fractional Hartley transform. These fractional transforms found number of applications in signal processing, image processing, quantum mechanics etc.

The Hilbert transform based on Fourier transform has widely been applied in many areas such as optical systems, modulation, edge detections [2, 4], etc. The Hilbert transformation is also used in the construction of analytic signals, which in turn can be used to construct the complex envelop of a real signal. The complex envelop method is very useful in finding the output of band pass filters [11].

The Hilbert transform of a signal $f(x)$ is defined as [11]

$$H[f(x)](t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x)}{t-x}\,dx$$

and the analytic part of $f(x)$ is defined as $F(t) = f(t) + iH[f](t)$.

One of the most important properties of analytic signals is that they contain no negative frequency components of the real signal. Zayed [11] defined the fractional Hilbert transform (FrHT) for the signal $f(x)$ with angle α as

$$H^{\alpha}[f(x)](t) = \tilde{f}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) K_{\alpha}(x,t)\,dx$$

where the kernel $K_{\alpha}(x,t) = \dfrac{e^{-i\frac{\cot\alpha}{2}t^2}\, e^{i\frac{\cot\alpha}{2}x^2}}{\pi(t-x)}$, $\quad t \in \mathbb{R},\ \alpha \neq 0, \frac{\pi}{2}, \pi,\ t \neq x$

when the integral exist, where the integral is a Cauchy Principal value.

Notice that generalized fractional Hilbert transform reduces to standard one as in [11] when $\alpha = \frac{\pi}{2}$.

In this paper first we define fractional Hilbert transform on the spaces of generalized functions in section two. Section three is devoted for proving Inversion and Uniqueness theorem for generalized fractional Hilbert transform. Lastly conclusion is given in section four.

II. GENERALIZED FRACTIONAL HILBERT TRANSFORM

For dealing fractional Hilbert transform in the generalized sense, first we define,

2.1 THE TESTING FUNCTION SPACE $E(\mathbb{R}^n)$

An infinitely differentiable complex valued function φ on $\mathbb{R}^n$ belongs to $E(\mathbb{R}^n)$ if for each compact set $K \subset S_\alpha$ where $S_\alpha = \left\{ x \in \mathbb{R}^n, |x| \le \alpha, \alpha > 0 \right\}$ and for $K \in \mathbb{R}^n$,

$$\gamma_{E,K}(\varphi) = \sup_{x \in K} \left| D^K \varphi(x) \right| < \infty$$

Clearly E is complete and so a Frechet space. Moreover, we say that f is a fractional Hilbert transformable if it is a member of E' (the dual space of E).

2.2 GENERALIZED FRACTIONAL HILBERT TRANSFORM ON $E'(\mathbb{R}^n)$

The generalized fractional Hilbert transform of $f(x) \in E'(\mathbb{R}^n)$ where $E'(\mathbb{R}^n)$ is the dual of the testing function space $E(\mathbb{R}^n)$, can be defined as

$$H^\alpha[f(x)](t) = \left\langle f(x), K_\alpha(x,t) \right\rangle \qquad \text{for each} \quad t \in \mathbb{R} \tag{1}$$

where $K_\alpha(x,t) = \dfrac{e^{-i\frac{\cot\alpha}{2}(t^2 - x^2)}}{\pi(t-x)}$, $\quad \alpha \ne 0, \dfrac{\pi}{2}, \pi$

The right hand side of (1) has meaning as the application of $f \in E'$ to $K_\alpha(x,t) \in E$. $H^\alpha[f(x)](t)$ is α^{th} order generalized fractional Hilbert transform of the function $f(t)$.

III. INVERSION THEOREM

In this section we have proved the inverse theorem in the generalized sense using Zemanian technique.

Theorem: Let $f(x) \in E'(\mathbb{R}^n)$, $0 \le x \le \pi$ s. t. $\alpha \ne 0, \dfrac{\pi}{2}, \pi$

supp $f \subset S_a : S_a = \left\{ x : x \in \mathbb{R}^n, |x| \le a, a > 0 \right\}$ and let $H^\alpha(t)$ be the generalized fractional Hilbert transform of *f* as defined by $H^\alpha[f(x)](t) = H^\alpha(t) = \left\langle f(x), K_\alpha(x,t) \right\rangle$

Then for each $\varphi \in D(I)$ we have

$$\lim_{r \to \infty} \left\langle \int_{-r}^{r} H_\alpha(t) \bar{K}_\alpha(x,t) dt, \varphi(x) \right\rangle = \left\langle f(x), \varphi(x) \right\rangle$$

where $\quad K_\alpha(x,t) = \dfrac{1}{\pi} \dfrac{e^{-i\frac{\cot\alpha}{2}(t^2 - x^2)}}{t - x}$ and $\bar{K}_\alpha(x,t) = \dfrac{1}{\pi} \dfrac{e^{i\frac{\cot\alpha}{2}(t^2 - x^2)}}{x - t}$

Proof: To prove the inversion theorem, we require the following lemmas to be used in the sequel.

3.1 Lemma: Let $H^\alpha[f(x)](t) = H^\alpha(t)$ for $0 \le \alpha \le \pi$ s. t. $\alpha \ne 0, \dfrac{\pi}{2}, \pi$ and supp

$f \subset S_a = \left\{ x : x \in \mathbb{R}^n, |x| \le a, a > 0 \right\}$ for $\varphi(x) \in D(I)$, set $\psi(t) = \displaystyle\int_{-\infty}^{\infty} \varphi(x) \bar{K}_\alpha(x,t) dx$

Then for any fixed number r, $-\infty < r < \infty$,

$$\int_{-r}^{r} \psi(t) \left\langle f(v), K_\alpha(v,t) \right\rangle d\tau = \left\langle f(v), \int_{-r}^{r} \psi(t)\, K_\alpha(v,t) d\tau \right\rangle, \quad (t = \sigma + i\tau) \tag{2}$$

where, $t \in C^n$ and *v* is restricted to a compact subset of $\mathbb{R}^n$.

Proof: The case $\varphi(x) = 0$ is trivial, so that consider $\varphi(x) \ne 0$, it can be easily seen that

$\int_{-r}^{r} \psi(t)\, K_\alpha(v,t)\, d\tau, \quad t = \sigma + i\tau$ is a C^∞ function of *v* and it belongs to E.

Hence the right hand side of (2) is meaningful.

To prove the equality, we construct the Riemann- sum for this integral and write

$$\int_{-r}^{r} \langle f(v), K_\alpha(v,t)\rangle\, \psi(t) d\tau = \lim_{m\to\infty} \sum_{n=-m}^{m-1} \langle f(v), K_\alpha(v, \sigma + i\tau_{n,m})\rangle\, \psi(\sigma + i\tau_{n,m})\Delta\tau_{n,m}$$

$$= \lim_{m\to\infty} \left\langle f(v), \sum_{n=-m}^{m-1} K_\alpha(v, \sigma + i\tau_{n,m})\, \psi(\sigma + i\tau_{n,m})\Delta\tau_{n,m} \right\rangle$$

Taking the operator D_v^k within the integral and summation signs, which is easily justified as,

$$\gamma_{K,k}\left\{ \sum_{n=-m}^{m-1} K_\alpha(v, \sigma + i\tau_{n,m})\, \psi(\sigma + i\tau_{n,m})\Delta\tau_{n,m} - \int_{-r}^{r} \psi(t) K_\alpha(v,t) d\tau \right\}$$

$$= \sup_{v\in K}\left\{ \sum_{n=-m}^{m-1} D_v^k K_\alpha(v, \sigma + i\tau_{n,m})\, \psi(\sigma + i\tau_{n,m})\Delta\tau_{n,m} - \int_{-r}^{r} \psi(t) D_v^k K_\alpha(v,t) d\tau \right\}$$

As $\lim_{m\to\infty} \sum_{n=-m}^{m-1} D_v^k (K_\alpha v, \sigma + i\tau_{n,m})\, \psi(\sigma + i\tau_{n,m})\Delta\tau_{n,m} = \int_{-r}^{r} \psi(t) D_v^k K_\alpha(v,t) d\tau \qquad \forall\ v \in K$

It follows that for every m, the summation is a member of E and it converges in E to the integral on the right side of (2).

3.2 Lemma: For $\varphi(t) \in D(I)$, set $\psi(t)$ as in lemma (3.1) above for $t \in C^n, v$ restricted to the subset of $\mathbb{R}^n$,

then $\quad M_r(v) = \int_{-r}^{r} \psi(t) K_\alpha(v,t) d\tau, \qquad t = \sigma + i\tau$

$$= \int_{-r}^{r} K_\alpha(v,t) d\tau \int_{-\infty}^{\infty} \varphi(t) \bar{K}_\alpha(x,t) d\tau \qquad (3)$$

converges in $E(\mathbb{R}^n)$, to $\varphi(v)$ as $r \to \infty$.

Proof: We shall show that $M_r(v) \to \varphi(v)$ in E as $r \to \infty$.

It is to show $\gamma_{K,k}[M_r(v) - \varphi(v)] = \sup_{v\in K} \left| D_v^k [M_r(v) - \varphi(v)] \right| \to 0$ as $r \to \infty$

We note that for $k = 0$,

$$\int_{-\infty}^{\infty} K_\alpha(v,t) \int_{-\infty}^{\infty} \varphi(x) \bar{K}_\alpha(x,t)\, dx d\tau = \varphi(v)$$

This is to say that $\lim_{r\to\infty} M_r(v) = \varphi(v)$

Since the integrand is a C^∞ function of *v* and $\varphi \in D(I)$, we can repeatedly differentiate under the integral sign in (3) and integrals are uniformly convergent, we have

$$\int_{-\infty}^{\infty} D_v^k K_\alpha(v,t) \int_{-\infty}^{\infty} \varphi(v) \bar{K}_\alpha(x,t)\, dx d\tau = \varphi(v) \qquad \forall\, v \in K$$

Hence the claim.

Now let $\varphi(x) \in D(I)$, we shall show that

$$\left\langle \int_{-r}^{r} \bar{K}_\alpha(x,t) H_\alpha(t) d\tau, \varphi(x) \right\rangle \text{ tends to } \langle f(x), \varphi(x)\rangle \text{ as } r \to \infty \qquad (4)$$

From the analyticity of $H^{\alpha}(t)$ on C^{∞} and the fact that $\varphi(x)$ has compact support in *I*, it follows that the left side expression in (4) is merely a repeated integral with respect to *x* and *t* and the integral in (4) is continuous function of *x* as the closed bounded domain of the integration.
Therefore, we can write (4) as

$$\int_{-\infty}^{\infty}\varphi(x)\int_{-r}^{r}\bar{K}_{\alpha}(x,t)H_{\alpha}(t)d\tau dx=\int_{-\infty}^{\infty}\varphi(x)\int_{-r}^{r}\bar{K}_{\alpha}(x,t)\langle f(v),K_{\alpha}(v,t)\rangle d\tau dx$$

$$=\int_{-r}^{r}\langle f(v),K_{\alpha}(v,t)\rangle\int_{-\infty}^{\infty}\varphi(x)\bar{K}_{\alpha}(x,t)dxd\tau$$

Since $\varphi(x)$ is compact support and the integrand is a continuous function of (x,t), the order of integration may be changed. The changed in the order of integration is justified by appeal to lemma 3.1.
This yield,

$$=\int_{-r}^{r}\langle f(v),K_{\alpha}(v,t)\rangle\psi(t)d\tau$$

where $\psi(t)$ is an in Lemma 3.1. Hence by Lemma 3.1, this is equal to $\left\langle f(v),\int_{-r}^{r}K_{\alpha}(v,t)\psi(t)d\tau\right\rangle$

that is

$$\int_{-r}^{r}\langle f(v),K_{\alpha}(v,t)\rangle\int_{-\infty}^{\infty}\varphi(x)\bar{K}_{\alpha}(x,t)dx\,d\tau=\left\langle f(v),\int_{-r}^{r}K_{\alpha}(v,t)\varphi(t)\,d\tau\right\rangle \qquad (5)$$

Again by lemma 3.1 equation (5) converges to $\langle f(x),\varphi(x)\rangle$ as $r\to\infty$.
This completes the proof of the theorem.
Proof of Inversion Theorem:
Let $\varphi\in D$, we have to show that

$$\lim_{r\to\infty}\left\langle\int_{-r}^{r}H^{\alpha}(t)\bar{K}_{\alpha}(x,t)dt,\varphi(x)\right\rangle=\langle f(x),\varphi(x)\rangle$$

The integral on t is a continuous function of x and therefore the left hand side without the limit notation can be written as

$$=\int_{-\infty}^{\infty}\varphi(x)\int_{-r}^{r}\bar{K}_{\alpha}(x,t)\langle f(v),K_{\alpha}(v,t))\rangle d\tau dx,\qquad s=\sigma+i\tau,\ r\to\infty$$

Since $\varphi(x)$ is of bounded support and the integrand is a continuous function of (x,t) the order of integration may be changed. This yield

$$=\int_{-r}^{r}\langle f(v),K_{\alpha}(v,t)\rangle\int_{-r}^{r}\varphi(x)\bar{K}_{\alpha}(x,t)dxd\tau$$

$$=\int_{-r}^{r}\langle f(v),K_{\alpha}(v,t)\rangle\Psi(t)d\tau$$

where $\Psi(t)$ is in Lemma 3.1, hence by Lemma 3.1.

$$=\int_{-r}^{r}\langle f(v),K_{\alpha}(v,t)\Psi(t)\rangle d\tau$$

$=\langle f(x),\varphi(x)\rangle$ as $r\to\infty$ using lemma 3.2.
This completes the proof of the theorem.

3.3. UNIQUENESS THEOREM

If $H^{\alpha}[f(x)](t)$ and $H^{\alpha}[g(x)](t)$ are generalized fractional Hilbert transform of *f(x)* and *g(x)* respectively such that $\alpha \neq 0, \frac{\pi}{2}, \pi$ and supp $f \subset S_a : S_a = \left\{x : x \in \mathbb{R}^n, |x| \leq a, a > 0\right\}$ and supp $g \subset S_b : S_b = \left\{x : x \in \mathbb{R}^n, |x| \leq b, b > 0\right\}$ if $H^{\alpha}[f(x)](t) = H^{\alpha}[g(x)](t)$ then $f = g$ in the sense of equality of $D'(I)$.

Proof: By Inversion theorem

$$f - g = \lim_{r\to\infty} \int_{-r}^{r} \left\{H^{\alpha}[f(x)](t) - H^{\alpha}[g(x)](t)\right\} \bar{K}_{\alpha}(x,t)\, dt$$

$$= 0$$

Thus $f = g$ in $D'(I)$.

where $\bar{K}_{\alpha}(x,t) = \frac{1}{\pi} \frac{e^{i\frac{\cot\alpha}{2}(t^2 - x^2)}}{x - t}$.

IV. CONCLUSION

In this paper fractional Hilbert transform is extended to the distribution of compact support by using the kernel method. Since fractional Hilbert transform is useful in signal processing tool, we believe that generalized fractional Hilbert transform will also become useful tool for signal processing and image processing particularly when the signals are impulse type.

REFERENCES

[1] Akay. O., Boudreaux-Bartels F., Fractional Mellin Transformation: an extension of fractional frequency concept for scale, 8th IEEE, Dig. Sign. Proc. workshop, Bryce, Canyan, Utah. 1998.

[2] Gabor D.: "Theory of communications," J. Inst. Elect. Eng., Vol. 93, pp. 429-457, Nov. 1946.

[3] Gudadhe A. S., Joshi A. V.: "Generalized half Linear Canonical Transform and Its Properties", International Journal of engineering Research and Technology", Vol. 2, Issue 6, June-2013.

[4] Kohalman K.: "Corner detection in natural images based on the 2-d Hilbert transform", Signal Process, Vol. 48, pp. 225-234, Feb. 1996.

[5] Namias V: The fractional Fourier transform and its application to quantum mechanics, J. Inst. Math. Appl., vol. 25, pp. 241–265, 1980.

[6] Sontakke, Gudadhe A. S.: "On Generalized fractional Hankel transform", Int. Journal of Math. Analysis, Vol. 6, No. 18, pp. 883 – 896, 2012.

[7] Tatiana Alieva and Bastiaans Martin J.: "On Fractional Fourier transform moments", IEEE Signal processing Letters, Vol. 7, No. 11, Nov. 2000.

[8] Tatiana Alieva and Bastiaans Martin J.: "Wigner distribution and fractional Fourier transform for 2- dimensional symmetric beams", JOSA A, Vol. 17, No. 12, Dec. 2000, pp. 2319-2323.

[9] Zayed A I: "A Convolution and Product theorem for the fractional Fourier transform", IEEE signal processing letters, Vol. 5, No. 4, April 1998.

[10] Zayed A. I.: "Function and Generalized Function Transformations, CRC press, Boca Raton, FL, 1996.

[11] Zayed A I: "Hilbert Transform Associated with the Fractional Fourier Transform", IEEE signal processing letters, Vol. 5, No. 8, Aug 1998.