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Design of Second Order Digital Differentiator and Integrator Using Forward Difference Formula and Fractional Delay

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ABSTRACT-*In this paper, the second order differentiator and integrator, design is investigated. Firstly, the forward difference formula is applied in numerical differentiation for deriving the transfer function of second order differentiator and integrator. Thereafter, the Richardson extrapolation is used for generating the high accuracy results, while using low order formulas. Further, the conventional Lagrange FIR fractional delay filter is applied directly for implementation of the second order differentiator, design. Finally, the effectiveness of this new design approach is illustrated by using several numerical examples.*

KEYWORDS -*Digital differentiator, Digital integrator, Forward difference formula, Fractional delay filter and Richardson extrapolation.*

I. INTRODUCTION

Digital differentiator and integrator are very powerful tools for the determination and estimation of time derivatives for given signals. For example, in radars, the second order differentiator is used for position measurement to compute the acceleration [1]. In image processing, Laplacian operator can be used to detect the edge of an image [2].In biomedical engineering, very often the higher order derivatives of medical data are required [3]. Several methods such aseigen filter methods are developed to design second order differentiator [4] and limit computation [5]. The ideal frequency response for second order differentiator and integrator is:

$$D_d(\omega) = (j\omega)^2 \tag{1}$$

$$D_I(\omega) = \frac{1}{(j\omega)^2} \tag{2}$$

Now, the challenge remains to design a digital filter so that the frequency response fits $D(\omega)$, very well. During the application of difference formulas for obtaining the transfer functions of second order differentiators, the design accuracy is not at-par. Further, for obtaining more accuracy in results, the simple algebraic procedure named, Richardson extrapolation is applied iteratively. The Richardson extrapolation has a historical background [6], owing to its proposal in 1927. So far, the computation of accurate bifurcation values of periodic responses [7] is done through extrapolation procedure. Additionally, the solution of paraxial wave equation [8] is obtained through extrapolation procedure. In addition to it, the improvement of accuracy of one sided finite difference formula is done through extrapolation. In this paper, the forward difference formula is applied for the derivation of the transfer function for second order differentiator. Thereafter, the Richardson extrapolation is used for generation of high accuracy results, using low order formulas. Though, there are fractional delay elements, involved with the transfer function of the designed differentiator, the Lagrange FIR fractional delay filter is applied directly for the implementation of the designed differentiator [10].

The integrators and differentiators are largely used for filters in which delay is inherent. Further, as the delay is responsible for reduced efficiency and accuracy so an improved design is suggested for the improvement in accuracy and efficiency. In this paper, Section II presents the design methods, Section III shows the proposed work and Section IV illustrates the Simulation Results and conclusions.

II. DESIGN METHOD

Given the signal$x(n)$, the forward difference formula to estimate its second order derivative x''(n) is:

$$x''(n) = \frac{x(n) - 2x(n+\alpha) + x(n+2\alpha)}{\alpha^2} \tag{3}$$

Taking Z-transform both the sides:

$$X''(z) = \frac{1 - 2z^{\alpha} + z^{2\alpha}}{\alpha^2} X(z) \tag{4}$$

Thus, the transfer function of differentiator is given by:

$$A_0(z,\alpha) = \frac{X''(z)}{X(z)} = \frac{1 - 2z^{\alpha} + z^{2\alpha}}{\alpha^2} \tag{5}$$

If z is replaced by $e^{j\omega}$, the frequency response is:

$$A_0(e^{j\omega},\alpha) = \frac{1-2e^{j\omega\alpha} + e^{2j\omega\alpha}}{\alpha^2} \tag{6}$$

Using the Taylor's series, exponential function can be expanded as:

$$e^{\theta} = \sum_{k=0}^{\infty} \frac{Q^k}{k!} \tag{7}$$

Then the frequency response will be:

$$A_0(e^{j\omega},\alpha) = \frac{1-2\sum_{k=0}^{\infty}\frac{(j\omega\alpha)^k}{k!} + \sum_{k=0}^{\infty}\frac{(2j\omega\alpha)^k}{k!}}{\alpha^2} \tag{8}$$

Further, simplifying the equation (8):

$$A_0(e^{j\omega},\alpha) = \frac{1-2\left\{1+\frac{j\omega\alpha}{1!}+\frac{(j\omega\alpha)^2}{2!}+\cdots\right\}}{\alpha^2} + \frac{\left\{1+\frac{2(j\omega\alpha)}{1!}+\frac{4(j\omega\alpha)^2}{2!}+\cdots\right\}}{\alpha^2} \tag{9}$$

$$A_0(e^{j\omega},\alpha) = \frac{\left[\frac{2(j\omega\alpha)^2}{2!}+\frac{6(j\omega\alpha)^3}{3!}+\frac{14(j\omega\alpha)^4}{2!}\cdots\right]}{\alpha^2} \tag{10}$$

$$A_0(e^{j\omega},\alpha) = (j\omega)^2\left[1+(j\omega\alpha)+\frac{7}{12}(j\omega\alpha)^2+\cdots\right] \tag{11}$$

$$A_0(e^{j\omega},\alpha) = (j\omega)^2 + \alpha(j\omega)^3 + \frac{7}{12}\alpha^2(j\omega)^4\cdots \tag{12}$$

$$A_0(e^{j\omega},\alpha) = D(\omega) + \sum_{k=1}^{\infty} a_k\alpha^k \tag{13}$$
$$= D(\omega) + O(\alpha)$$

Where notation $O(\alpha)$ means error term decays as fast asα. If parameter α approaches to zero then:

$$\lim_{\alpha\to 0} A_0(e^{j\omega},\alpha) = D(\omega)$$

That is $A_0(z,\alpha)$ approaches to ideal differentiator when parameter α tends to zero. However, as the convergence speed is not good enough so Richardson extrapolation is used for generation of high accuracy results. Two expressions from Equation (13):

$$A_0(e^{j\omega},\alpha) = D(\omega) + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \cdots \tag{15}$$
$$A_0(e^{j\omega},2\alpha) = D(\omega) + 2a_1\alpha + 4a_2\alpha^2 + 8a_3\alpha^3 + \cdots \tag{16}$$

If we multiple equation (15) by 2 and subtract(16):

$$2A_0(e^{j\omega},\alpha) - A_0(e^{j\omega},2\alpha) = D(\omega) + 0 - 2a_1\alpha - 6a_2\alpha^2\cdots \tag{17}$$

After some manipulation the equation is rewritten as:

$$A_1(e^{j\omega},\alpha) = 2A_0(e^{j\omega},\alpha) - A_0(e^{j\omega},2\alpha)$$
$$= D(\omega) - 2a_1\alpha - 6a_2\alpha^2\cdots$$
$$= D(\omega) + \sum_{k=2}^{\infty} b_k\alpha^k \tag{18}$$
$$= D(\omega) + O(\alpha^2)$$

Where coefficients b_k are given by:

$$b_k = (2-2^k)a_k \tag{19}$$

The order of error term of $A_1(e^{j\omega},\alpha)$ is $O(\alpha^2)$, which results in faster convergence speed than $A_0(e^{j\omega},\alpha)$, when α approaches zero. Replacing $e^{j\omega}\,byz$ and using Equation (5), Equation (19), reduce to the following form:

$$A_1(e^{j\omega},\alpha) = 2A_0(e^{j\omega},\alpha) - A_0(e^{j\omega},2\alpha)$$
$$= 2\left(\frac{1-2z^{\alpha}+z^{2\alpha}}{\alpha^2}\right) - \left(\frac{1-2z^{2\alpha}+z^{4\alpha}}{\alpha^2}\right)$$
$$= \frac{7-16z^{\alpha}+10z^{2\alpha}-z^{4\alpha}}{4\alpha^2} \tag{20}$$

So far we have already seen the usage of parameter α and 2α for removing the error term involving α. Looking at the way α^2 is removed from two expressions. Equation (19) are given by:

$$A_1(e^{j\omega},\alpha) = D(\omega) + b_2\alpha^2 + b_3\alpha^3 + \cdots \tag{21}$$
$$A_1(e^{j\omega},2\alpha) = D(\omega) + 4b_2\alpha^2 + 8b_3\alpha^3 + \cdots \tag{22}$$

If we multiply Equation (21) by 4 and subtract Equation (22) from this product then the terms involving d_2 cancel and the result is:

$$4A_1(e^{j\omega},\alpha) - A_1(e^{j\omega},2\alpha) = 3D(\omega) + 0 - 4b_3\alpha^3 \cdots \tag{23}$$

On algebraic simplification:

$$A_2(e^{j\omega},\alpha) = \frac{4A_1(e^{j\omega},\alpha) - A_1(e^{j\omega},2\alpha)}{3}$$
$$= D(\omega) + 0 - \frac{4}{3}b_3\alpha^3 - \frac{4}{3}b_4\alpha^4 + \cdots \tag{24}$$
$$= D(\omega) + O(\alpha^3)$$

Clearly the order of error term of $A_2(e^{j\omega},\alpha)$ is $O(\alpha^3)$. Replacing $e^{j\omega}$ by z and using Equation (20) then Equation (24) becomes:

$$A_2(z,\alpha) = \frac{4A_1(z,\alpha) - A_1(z,2\alpha)}{3}$$
$$= \frac{105 - 256z^{\alpha} + 176z^{2\alpha} - 26z^{4\alpha} + z^{8\alpha}}{48\alpha^2} \tag{25}$$

Based on the above result of forward difference, the general recursive formula for Richardson process is stated below:

$$A_k(z,\alpha) = \frac{2^k A_{k-1}(z,\alpha) - A_{k-1}(z,2\alpha)}{2^k - 1} \tag{26}$$

Then the frequency response has the form:

$$A_k(e^{j\omega},\alpha) = D(\omega) + O(\alpha^{k+1}) \tag{27}$$

Moreover, from Equation (5), (20) and Equation (25), it is easy to observe that the following equality is always valid:

$$A_k(z,2\alpha) = \frac{1}{4}A_k(z^2,\alpha) \tag{28}$$

Based on this equality, a recursive computation method is developed to get $A_k(z,\alpha)$ can be rephrased in the unified form:

$$A_k(z,\alpha) = r_0^{(k)} + \sum_{m=0}^{k+1} r_{m+1}^{(k)}\, z^{2^{m\alpha}} \tag{29}$$

For $k = 0,1,2$, the coefficients $r_m^{(k)}$ are:

$$k = 0:\ r_0^{(0)} = \frac{1}{\alpha^2}, r_1^{(0)} = \frac{-2}{\alpha^2}, r_2^{(0)} = \frac{1}{\alpha^2}$$
$$k = 1:\ r_0^{(1)} = \frac{7}{4\alpha^2}, r_1^{(1)} = \frac{-16}{4\alpha^2}, r_2^{(1)} = \frac{10}{4\alpha^2}, r_3^{(1)} = \frac{-1}{4\alpha^2}$$
$$k = 2:\ r_0^{(2)} = \frac{105}{48\alpha^2}, r_1^{(2)} = \frac{-256}{48\alpha^2}, r_2^{(2)} = \frac{176}{48\alpha^2}, r_3^{(2)} = \frac{-26}{48\alpha^2}, r_4^{(2)} = \frac{1}{48\alpha^2}$$

The coefficients of differentiator $A_k(z,\alpha)$ can be obtained. To evaluate the performance of differentiator$A_k(z,\alpha)$, the normalized root- mean square (NRMS) error is defined by:

$$E(\alpha) = \left(\frac{\int_0^{\pi}|A_k(z,\alpha) - D(\omega)|^2 d\omega}{\int_0^{\pi}|D(\omega)|^2\, d(\omega)}\right)^{\frac{1}{2}} \times 100\% \tag{30}$$

The smaller the NRMS error E (α), the better the digital differentiator's performance for values of k=0, 1, 2. It is quite evident that the error can be reduced by decreasing α or increasing k. The graph in figure 1 is a clear representation of normalized root mean square error response for three different cases. In the three cases value of k is changed from 0 to 2, at k = 0, it is clear that response of graph in terms of error versus frequency is high and quite linear but when value of k is either 1 or 2, then error is low at low frequencies thus suggesting the user to use a system with low values of frequency.

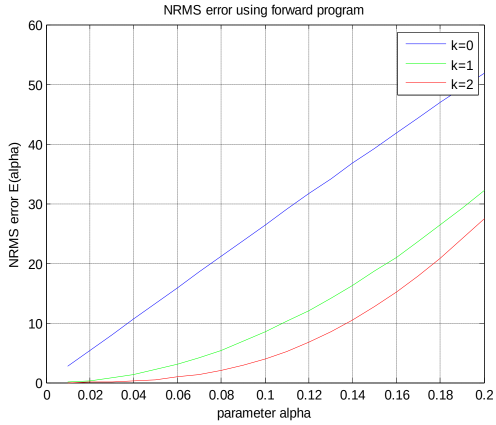

Fig. 1 The NRMS error curves $E(\alpha)$ of the designed second order differentiator for k=0, 1, 2

Fig. 2 illustrates the frequency response error $20\log_{10}\left(\left|A_k(e^{j\omega},\alpha)-D(\omega)\right|\right)$ for $\alpha = 0.1$ and it is a copy of frequency response error of a differentiator at three different values of k =0, 1 and 2. At times when frequency is high the error is saturating at a constant value but with low frequency values the error surge is quite high.

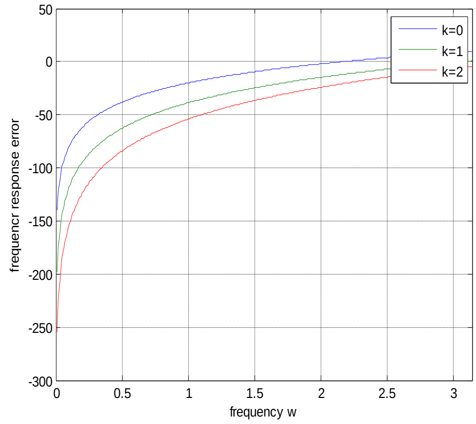

Fig. 2 The frequency response error $20\log_{10}\left(\left|A_k(e^{j\omega},\alpha)-D(\omega)\right|\right)$ for $\alpha = 0.1$ and $k = 0, 1, 2$

Further, the error at the complete frequency range is gets reduced with the increase in k. As the implementation of differentiator $A_k(z,\alpha)$involves fractional delay, this problem can be solved by Lagrange fractional delay filters. For, achieving it, a pure integer delay z^{-1} is cascaded with $A_k(z,\alpha)$ in Equation. (29) to get the transfer function of linear phase differentiator:

$$B_k(z,\alpha) = z^{-1}A_k(z,\alpha) \tag{31}$$

$$= r_0^{(k)}z^{-I} + \sum_{m=0}^{k+1} r_{m+1}^{(k)} z^{-I-2^m\alpha}$$

The differentiator $B_k(z,\alpha)$ approximates the ideal frequency response, $D(\omega)$.

The fractional delay is approximated by:

$$z^{-(I+p)} \approx \sum_{n=0}^{L} h_n(p) z^{-n} \tag{32}$$

Where filter coefficients $h_n(p)$ is represented in the explicit form:

$$h_n(p) = \prod_{\substack{k=0 \\ k\neq n}}^{L} \frac{I + 2\alpha - k}{n-k} \tag{33}$$

If we substitute the Equation (32) into Equation (31) and choose $p = 2^m\alpha$, the designed second order differentiator is:

$$B_k(z,\alpha) = r_0^{(k)} z^{-I} + \sum_{m=0}^{k+1} r_{m+1}^{(k)} \left(\sum_{n=0}^{L} h_n(2^m\alpha)\, z^{-1} \right)$$

$$= \sum_{n=0}^{L} \beta_k(n) z^{-n} \quad (34)$$

Where coefficients are:

$$\beta_k(n) = r_0^{(k)} \delta(n-1) + \sum_{m=0}^{k+1} r_{m+1}^{(k)} h_n(2^m \alpha) \quad (35)$$

The notation $\delta(.)$denotes the delta function. If we substitute the Equation (33) into Equation (35), the coefficients for k=0, 1, 2 is written as:

$$\beta_0 = \frac{1}{\alpha^2} \delta(n-1) - \frac{2}{\alpha^2} \prod_{\substack{k=0 \\ k \neq n}}^{L} \frac{I + \alpha - k}{n-k} + \frac{1}{\alpha^2} \prod_{\substack{k=0 \\ k \neq n}}^{L} \frac{I + 2\alpha - k}{n-k} \quad (36)$$

$$\beta_1 = \frac{7}{4\alpha^2} \delta(n-1) - \frac{16}{4\alpha^2} \prod_{\substack{k=0 \\ k \neq n}}^{L} \frac{I + \alpha - k}{n-k} + \frac{10}{4\alpha^2} \prod_{\substack{k=0 \\ k \neq n}}^{L} \frac{I + 2\alpha - k}{n-k} - \frac{1}{4\alpha^2} \prod_{\substack{k=0 \\ k \neq n}}^{L} \frac{I + 4\alpha - k}{n-k} \quad (37)$$

Now an example with $\propto$= 0.1 and L= 80 is used for illustrating the design. The frequency response error for I=36 and k=0, 1 is illustrated by the $20\log_{10}\left(\left|B_k(e^{j\omega}, \alpha) - e^{-j\omega I} D(\omega)\right|\right)$ in Fig 3. It is quite obvious that these error curves are similar to the curves in Fig 2, except some high frequency range, distortions. It is due to the maximally flat design of Lagrange fractional delay at frequency $\omega = 0$. Further, the frequency response calculation is imperative and ostensible while working on an integrator or differentiator but the response of both systems has almost similar response at high frequencies, though at low frequency values, the response is increased.

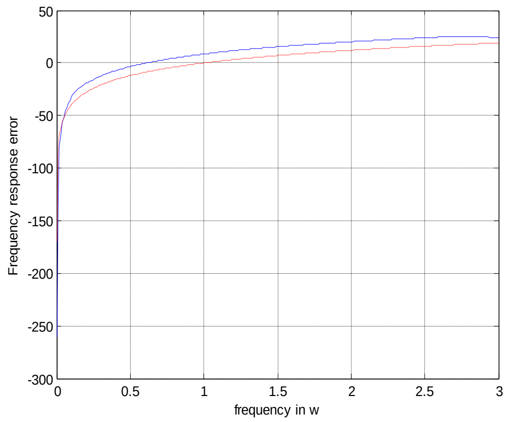

Fig. 3 The frequency response error $20\log_{10}\left(\left|B_k(e^{j\omega}, \alpha) - e^{-j\omega I} D(\omega)\right|\right)$for $\alpha = 0.1$and $k = 0, 1,$ $and\, I = 36$.

Figure 4 has an indicative response of magnitude versus frequency which helps in creating a similar surge in two different magnitude responses when measured at two different values of I =36 and 40.

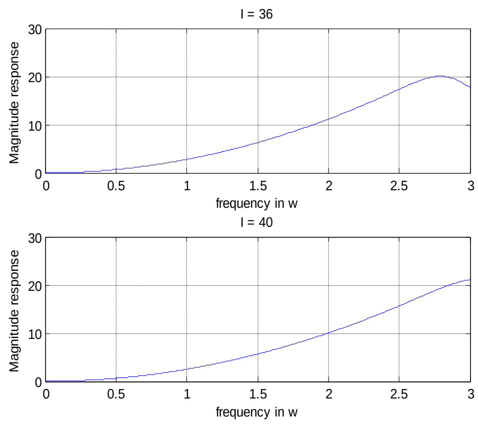

Fig. 4 The magnitude response of the designed second order differentiator error $B_0(z, \alpha)$,$(a) I = 36$, $(b)\, I = 40$

III. PRESENTED WORK

This paper has described the previous work related to differentiator in digital domain, using z transformers. Further, the research work was eventually, extended to integrator. The design and development of integrator is carried out by utilizing the techniques such as Richardson extrapolation, as discussed in K Rahul S. N. Bhattacharaya [9] and Simpsons Rule for Integration, which has relation of taking steps in the integrator, as discussed in M. A. Al-Alaoui [13].

This paper removes the steps concept from integrator by considering the equation of differentiator and inversing it and thereafter, adjusting the parameters of equation for representing integrator. The basic continuous representation of integration is represented in equation 2. Further, the equation 6 is inversed for obtaining the frequency response of integrator:

$$A_0(e^{j\omega},\alpha) = \frac{\alpha^2}{1 - 2e^{j\omega\alpha} + e^{2j\omega\alpha}} \tag{38}$$

For value, $A_1(e^{j\omega},\alpha)$, the inverse is represented as:

$$A_1(e^{j\omega},\alpha) = \frac{4\alpha^2}{7 - 16e^{j\omega\alpha} + 10e^{2j\omega\alpha} - e^{4j\omega\alpha}} \tag{39}$$

For value, $A_2(z,\alpha)$

$$A_2(e^{j\omega},\alpha) = \frac{48\alpha^2}{105 - 256e^{j\omega\alpha} + 176e^{2j\omega\alpha} - 26e^{4j\omega\alpha} + e^{8j\omega\alpha}} \tag{40}$$

Fig. 5 illustrates the frequency response error $20\log_{10}\left(\left|A_k(e^{j\omega},\alpha) - D(\omega)\right|\right)$ for $\alpha = 0.1$ At the system level integrator will have a similar error when the response is just a inverse of equation no matter what values of k are selected k =1 act as a threshold for two different integrators when the response of error has a constant magnitude at high frequency.

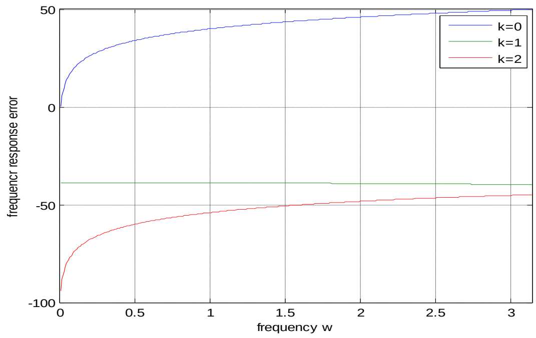

Fig. 5 The frequency response error $20\log_{10}\left(\left|A_k(e^{j\omega},\alpha) - D(\omega)\right|\right)$ for $\alpha = 0.1$and $k = 0, 1,\ 2$

Now an example with ∝= 0.1 and L= 80 is used for illustrating the design. The frequency response error for I=36 and k=0, 1 is illustrated by the $20\log_{10}\left(\left|B_k(e^{j\omega},\alpha) - e^{-j\omega I}D(\omega)\right|\right)$ in Fig 6. Further, the delay in differentiator is a common response but with increasing values of frequency this delay will have a negative impact. The above two different lines in figure 6 have delay error response with frequency for a delay system and both of them; depict a high surge of error in the graph.

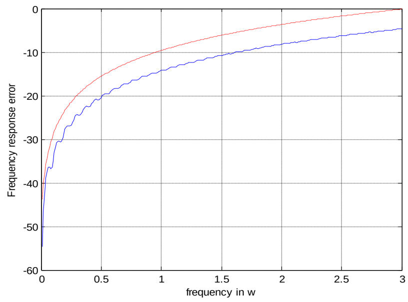

Fig. 6 The frequency response error $20\log_{10}\left(\left|B_k(e^{j\omega},\alpha) - e^{-j\omega I}D(\omega)\right|\right)$for $\alpha = 0.1$and $k = 0, 1,\ $ and $I = 36.$

Figure 7 is a linear plot of magnitude response for second order differentiator, with respect to frequency. The increasing response, with increase in frequency is good however here the magnitude response of differentiator increases indefinitely with little increase in smaller frequency values which shows less feasibility of implementation of the filters.

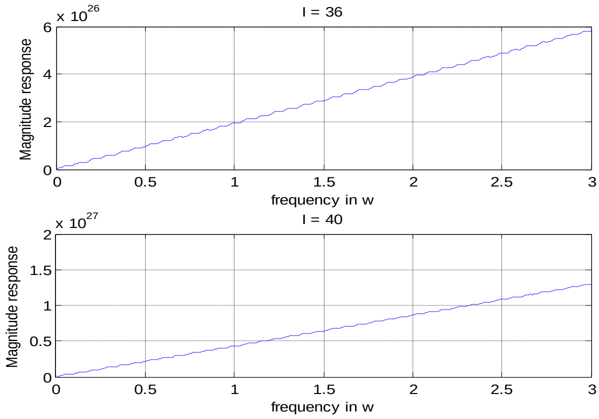

Fig. 7 The magnitude response of the designed second order integrator error $B_0(z,\alpha)$, $(a) I = 36$, $(b) I = 40$

IV. CONCLUSION

The paper includes the steps, taken for improving the efficiency of the digital differentiators. Additionally, the numerical techniques can be used for finding better solutions which can result in stable magnitude response and higher optimization value. Further, the techniques illustrated in the paper work, show the derivation of transfer functions. In the dissertation, the design of second order differentiator and integrator is illustrated. Firstly, the forward difference formula is used for generating the transfer function. Thereafter, the Richardson extrapolation is used to generate high accuracy results, while low order formulas are used. The Lagrange filter is applied for the implementation of the designed differentiator and integrator.

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