Fixed Point Theorem in Fuzzy Metric Space Using (CLRg) Property

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ABSTRACT: The object of this paper is to establish a common fixed point theorem for semi-compatible pair of self maps by using CLRg Property in fuzzy metric space.

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I. INTRODUCTION


II. PRELIMINARIES

Example 2.1 [6] Let (X,d) be a metric space. Define a*b = min{a,b} and

\[ M(x,y,t) = \frac{t}{t + d(x,y)} \]

for all \( x, y \in X \) and all \( t > 0 \). Then \( (X,M,*) \) is a fuzzy metric space.

A sequence \( \{x_n\} \) is said to be a Cauchy sequence if \( \lim_{n \to \infty} M(x_n+p,x_n,t) = 1 \) for any \( t > 0 \) and for each \( p > 0 \).

A fuzzy metric space \((X, M,*)\) is Complete if every Cauchy sequence in \( X \) converge to \( X \).

Example 2.2 [11] Let \( X \) be any set. A fuzzy set \( A \) in \( X \) is a function with domain \( X \) and Values in \([0,1]\).

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A Binary operation \(*: [0,1] \times [0,1] \rightarrow [0,1]\) is called a continuous t-norms if an topological monoid with unit 1 such that \( a*b \leq c*d \) whenever \( a \leq c \) and \( b \leq d \), for all \( a,b,c,d \in [0,1] \).

Examples of t-norms are \( a*b = ab \) and \( a*b = \min \{a,b\} \).

Definition 2.2 [6] The triplet \((X,M,*\) is said to be a Fuzzy metric space if \( X \) is an arbitrary set, \(*\) is a continuous t-norm and \( M \) is a fuzzy set on \( X \times (0,\infty) \) satisfying the following conditions; for all \( x,y,z \in X \) and \( t,s > 0 \),

\[
\begin{align*}
& (i) \quad M(x,y,0) = 0, \quad M(x,y,t) > 0, \\
& (ii) \quad M(x,y,t) = 1, \quad \text{for all } t > 0 \text{ if and only if } x = y, \\
& (iii) \quad M(x,y,t) = M(y,x,t), \\
& (iv) \quad M(x,y,t) * M(y,z,s) \leq M(x,z,t+s), \\
& (v) \quad M(x,y,t): [0,\infty) \rightarrow [0,1] \text{ is left continuous.}
\end{align*}
\]

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for all \( x, y \in X \) and all \( t > 0 \). Then \((X,M,*)\) is a fuzzy metric space.

A sequence \( \{x_n\} \) in a fuzzy metric space \((X,M,*)\) is said to be a Cauchy sequence if \( \lim_{n \to \infty} M(x_{n+p},x_n,t) = 1 \) for every \( t > 0 \) and for each \( p > 0 \).

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A sequence \( \{x_n\} \) in a fuzzy metric space \((X,M,*)\) is said to be Convergent to \( x \) in \( X \) if \( \lim_{n \to \infty} M(x_n,X,t) = 1 \), for each \( t > 0 \).

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A fuzzy metric space \((X,M,*)\) is Complete if every Cauchy sequence in \( X \) converge to \( X \).

Two self mappings \( P \) and \( Q \) of a fuzzy metric space \((X,M,*)\) are said to be Compatible if \( \lim_{n \to \infty} M(PQ(x_n),Qx_n,t) = 1 \) whenever \( \{x_n\} \) is a sequence such that \( \lim_{n \to \infty} Qx_n = z \) for some \( z \in X \).

Definiton 2.5 [13] Self maps \( A \) and \( S \) of a fuzzy metric space \((X,M,*)\) are said to be Weakly Compatible if they commute at their coincidence points. if, \( AP = SP \) for some \( p \in X \) then \( AP = SP \).
Lemma 2.1 [8] Let \( \{ y_n \} \) be a sequence in an FM-space \( X \). If there exists a positive number \( k \leq 1 \) such that \( M(y_{n+2}, y_{n+1}, t) \geq M(y_{n+1}, y_n, t)\), \( \forall t \in [0,1]\), then \( \{ y_n \} \) is a Cauchy sequence in \( X \).

Lemma 2.2 [8] If for two points \( x, y \) in \( X \) and a positive number \( k \leq 1 \) \( M(x, y, t) \geq M(x, y_0, t) \), then \( x = y \).

Lemma 2.3 [14] For all \( x, y \in X \), \( M(x, y, t) \) is a non-decreasing function.

Definition 2.10 [8] A pair \( (A, S) \) of self maps of a fuzzy metric space \( (X, M, \ast) \) is said to be semi compatible if \( \lim_{n \to \infty} M(Ax_n, x, t) = \lim_{n \to \infty} M(Sx_n, x, t) \leq \varphi(t) \), whenever \( \{ x_n \} \) is a sequence such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \) for some \( x \in X \).

It follows that \( (A, S) \) is semi compatible and \( Ay = Sx \) then \( ASy = SAy \)

Example 2.2 Let \( X = [0, 1] \) and \( (X, M, \ast) \) be the induced fuzzy metric space with \( M(x, y, t) = \max \{ 0, 1, \frac{3}{2} \} \).

Define self maps \( P \) and \( Q \) on \( X \) as follows:

\[
P(x) = \begin{cases} 2 & \text{if } 0 \leq x \leq 1 \\ x/2 & \text{if } 1 < x \leq 2 \\ \end{cases}
\]

\[
Q(x) = \begin{cases} 2 & \text{if } x = 1 \\ x + 3/5 & \text{if } 1 < x \leq 2 \\ \end{cases}
\]

And \( x_n = 2 - 1/2^n \). Then we have \( P(1) = Q(1) = 2 \) and \( S(2) = A(2) = 1 \).

Let \( \{ y_n \} \) be a sequence in \( X \) such that

\[
\lim\sup_{n \to \infty} Px_n = 1 \quad \lim\sup_{n \to \infty} Qx_n = 1 \text{ for some } u \in X.
\]

Definition 2.10 [9] A pair of self mapping \( P \) and \( Q \) of a fuzzy metric space \( (X, M, \ast) \) is said to satisfy the \( (CLR_g) \) property if there exists a sequence \( \{ x_n \} \) in \( X \) such that

\[
\lim_{n \to \infty} M(PQx_n, y, t) = \lim_{n \to \infty} M(QPx_n, y, t).
\]

Definition 2.11 [9] Two pairs \( (A, S) \) and \( (B, T) \) of self mappings of a fuzzy metric space \( (X, M, \ast) \) are said to share \( CLR_g \) of \( S \) property if there exist two sequence \( \{ x_n \} \) and \( \{ y_n \} \) in \( X \) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z
\]

for some \( z \in S(X) \) and \( z \in T(X) \).

Proposition 2.1 [4] In a fuzzy metric space \( (X, M, \ast) \) limit of a sequence is unique.

Example 2.3 Let \( X = [0, \infty) \) be the usual metric space. Define \( g, h : X \to X \) by

\[
g(x) = x + 3 \quad \text{and} \quad h(x) = 4x.
\]

Since \( \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} h(x_n) = 4 = g(1) \in X \). Therefore \( g \) and \( h \) satisfy the \( (CLR_g) \) property.

Lemma 2.4 Let \( A, B, S \) and \( T \) be four self mapping of a fuzzy metric space \( (X, M, \ast) \) satisfying following:

1. The pair \( (A, S) \) (or \( (B, T) \)) satisfies the common limit in the range of \( S \) property (or \( T \) property)
2. There exists a constant \( k \in (0, 1) \) such that
   \[
   (M(Ax_n, By_n, t))^{\frac{1}{2}} \leq \min ((M(Sx, t))^{\frac{1}{2}}, (M(Sx, t))^{\frac{1}{2}}, (M(Ty, t))^{\frac{1}{2}}, (M(Ty, t))^{\frac{1}{2}}),
   \]
   \[
   M(Sx, t), M(Ty, t)).
   \]

3. \( A(X) \subseteq T(X) \) (or \( B(X) \subseteq S(X) \)).

Then the pairs \( (A, S) \) and \( (B, T) \) share the common limit in the range property.

Singh and Jain [8] proved the following results.

Theorem 2.1 Let \( A, B, S \) and \( T \) be self maps on a complete fuzzy metric space \( (X, M, \ast) \) satisfying:

1. \( A(X) \subset T(X), B(X) \subset T(X) \)
2. One of \( A \) and \( B \) is continuous.
3. \( (A, S) \) is semi compatible and \( (B, T) \) is weak compatible.
4. For all \( x, y \in X \) and \( t > 0 \)
   \[
   M(Ax, By, t) \geq \varphi(A, B, t).
   \]

Where \( \varphi : [0,1] \to [0,1] \) is a continuous function such that \( \varphi(t) > t \), for each \( 0 < t < 1 \). Then \( A, B, S \) and \( T \) have a unique common fixed point.
III. MAIN RESULT

In the following theorem we replace the continuity condition by using (CLRg) property.

**Theorem 3.1** Let A, B, S and T be self mapping on a complete fuzzy metric space \((X,M,*)\), where * is a continuous t-norm defined by \(ab = \min \{a,b\}\) satisfying

(i) \(A(X) \subseteq T(X)\), \(B(X) \subseteq S(X)\).

(ii) \((B,T)\) is semi compatible

(iii) For all \(x,y \in X\) and \(t > 0\).

\[ M(Ax, By, kt) \geq \phi \left[ \min \{ M(Ax, Ty, t) , M(Bx, Ax, t) \} \right], \]

\[ \frac{1}{2}( M(Ax, Ty, t) + M(Bx, Ax, t) ) \]

Where \(\phi : [0,1] \to [0,1]\) is a continuous function such that \(\phi(1) = 1\), \(\phi(0) = 0\) and \(\phi(b) > b\) for \(0 < b < 1\).

If the pair \((A,S)\) and \((B,T)\) share the common limit in the range of \(S\) property, then \(A, B, S\) and \(T\) have a unique common fixed point

**Proof** – Let \(x_0\) be any arbitrary point for which there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[ y_{2n+1} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+2} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 0,1,2,\ldots \]

Now, \(M(y_{2n+1}, y_{2n+2}, kt) = M(Ax_{2n}, Bx_{2n+1}, kt)\)

\[ \geq \phi \left[ \min \{ M(Tx_{2n}, x_{2n+1}, t) , M(Bx_{2n+1}, Tx_{2n+1}, t) \} \right], \]

\[ \frac{1}{2}( M(Tx_{2n}, x_{2n+1}, t) + M(Bx_{2n+1}, Tx_{2n+1}, t) ) \]

\[ \geq \phi \left[ \min \{ M(y_{2n}, y_{2n+1}, t) , M(y_{2n+2}, y_{2n+1}, t) \} \right], \]

\[ \frac{1}{2}( M(y_{2n}, y_{2n+1}, t) + M(y_{2n+2}, y_{2n+1}, t) ) \]

\[ M(y_{2n+1}, y_{2n+2}, kt) > M(y_{2n}, y_{2n+1}, kt) \]

Similarly, we can prove \(M(y_{2n+2}, y_{2n+3}, kt) > M(y_{2n+1}, y_{2n+2}, kt)\)

In general, \(M(y_{n+1}, y_n, kt) > M(y_n, y_{n+1}, kt)\)

Thus, from this we conclude that \(M(y_n, y_{n+1}, kt)\) is an increasing sequence of positive real numbers in \([0,1]\) and tends to limit \(l \leq 1\).

If \(l < 1\), then \(M(y_n, y_{n+1}, kt) \geq \phi(M(y_n, y_{n+1}, kt))\)

Letting \(n \to \infty\), we get \(\lim_{n \to \infty} M(y_n, y_{n+1}, kt) \geq \phi(\lim_{n \to \infty} M(y_n, y_{n+1}, kt))\)

\(l \geq \phi(l) = l\) (Since \(\phi(b) > b\) for \(0 < b < 1\)).

a contradiction. Now for any positive integer \(q\)

\[ M(y_{2n}, y_{n+1}, kt) \geq M(y_{2n}, y_{n+1}, y_{n+q}, t) / (2(q-1)+1) \]

Taking limit, we get

\[ \lim_{n \to \infty} M(y_{2n}, y_{n+1}, y_{n+q}, t) / (2(q-1)+1) \]

\(\lim_{n \to \infty} M(y_{2n}, y_{n+1}, y_{n+q}, t) / (2(q-1)+1) \geq l\)

Which means \(\{y_n\}\) is a Cauchy sequence in \(X\). Since \(X\) is complete, then \(y_n \to z\) in \(X\).

That is \(\{Ax_n\}, \{Tx_{2n+1}\}, \{Bx_{2n+1}\}\) and \(\{Sx_n\}\) also converges to \(z\) in \(X\).

Since, the pair \((A,S)\) and \((B,T)\) share the common limit in the range of \(S\) property, then there exist two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[ \lim_{n \to \infty} x_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = Sz, \text{ for some } z \in X. \]

First we prove that \(Az = Sz\)

By \(3.3\), putting \(x = z\) and \(y = y_n\), we get

\[ M(Az, By_n, kt) \geq \phi \left[ \min \{ M(Az, Ty_n, t) , M(Sz, Az, t) . M(By_n, Ty_n, t) \} \right], \]

\[ \frac{1}{2}( M(Az, Ty_n, t) + M(By_n, Az, t) ) \]

Taking limit \(n \to \infty\), we get

\[ M(Az, Sz, kt) \geq \phi \left[ \min \{ M(Sz, Sz, t) , M(Sz, Az, t) \} \right], \]

\[ \frac{1}{2}( M(Az, Sz, t) + M(Sz, Az, t) ) \]

\[ M(Az, Sz, kt) \geq M(Sz, Az, t) \]

Hence by Lemma 2.2, we get \(Az = Sz\) …(1)

Since, \(A(X) \subseteq T(X)\), therefore there exist \(u \in X\), such that \(Az = Tu\) …(2)

Again, by inequality (iii), putting \(x = z\) and \(y = u\), we get

\[ M(Az, Su, kt) \geq \phi \left[ \min \{ M(Sz, Tu, t) , M(Sz, Az, t) \} \right], \]

\[ \frac{1}{2}( M(Az, Tu, t) + M(Su, Az, t) ) \]

Using (1) and (2), we get

\[ M(Tu, Buk, t) \geq \phi \left[ \min \{ M(Az, Tu, t) , M(Su, Az, t) \} \right], \]

\[ \frac{1}{2}( M(Az, Tu, t) + M(Su, Az, t) ) \]
\[ \geq \emptyset \left[ \min M(Tu,Tu,t), \{ 1, M(Bu,Tu,t) \}, M(Bu,Tu,t) \right] \]

\[ M(Tu,Bu,kt) \geq M(Bu,Tu,t) \]

Hence, by Lemma 2.2, we get \( Tu=Bu \) \( \ldots(3) \)

Thus, from (1), (2) and (3), we get \( Az= Sz= Tu= Bu \) \( \ldots(4) \)

Now, we will prove that \( Az=z \)

By inequality (iii), putting \( x=z \) and \( y=x_{2n+1} \), we get

\[ M(Az,Bx_{2n+1},kt) \geq \emptyset \left[ \min (M(Sz,Tx_{2n+1},t), \{ M(Sz,Az,t) , M(Bx_{2n+1},Tx_{2n+1},t) \} \right] \]

\[ \frac{1}{2} (M(Az,Tx_{2n+1},t) + M(Bx_{2n+1},Az,t)) \]

Taking limit \( n \to \infty \), using (1), we get

\[ M(Az,z,t) \geq \emptyset \left[ \min (M(Az,z,t), \{ M(Az,Az,t) , M(z,z,t) \}, \frac{1}{2} (M(Az,z,t)+ M(Az,Az,t)) \right] \]

\[ M(Az,z,t) \geq \emptyset \left[ \min (M(Az,z,t), \{ M(Az,z,t) \}, \frac{1}{2} (M(Az,z,t)+ M(Az,Az,t)) \right] \]

\[ M(Az,z,t) \geq M(Az,z,t) \]

Hence, by Lemma 2.2, we get \( Az=z \)

Thus, from (4), we get \( z= Tu=Bu \)

Now, Semi compatibility of \( (B,T) \) gives \( BT_{2n+1} \to Tz \), i.e., \( Bz=Tz \).

Now, putting \( x=z \) and \( y= z_{2n+1} \), we get

\[ M(Az,Bz,t) \geq \emptyset \left[ \min (M(Sz,Tz_{2n+1},t), \{ M(Sz,Az,t) , M(Bz,Tz_{2n+1},t) \}, \frac{1}{2} (M(Az,Tz_{2n+1},t) + M(Bz,Az,t)) \right] \]

\[ \frac{1}{2} (M(Az,Tz_{2n+1},t) + M(Bz,Az,t)) \]

\[ M(Az,Bz,t) \geq M(Az,Bz,t) \]

Hence, by Lemma 2.2, we get \( Az=Bz \).

And, hence \( Az= Bz= z \).

Combining all result we get \( z= Az=Bz=Sz=Tz \).

From this we conclude that \( z \) is a common fixed point of \( A,B,S \) and \( T \).

**Uniqueness**

Let \( z_1 \) be another common fixed point of \( A,B,S \) and \( T \). Then

\[ z_1 = Az_1 = Bz_1 = Sz_1 = Tz_1 \text{, and } z = Az = Bz = Sz = Tz \]

Then, by inequality (iii), putting \( x=z \) and \( y=z_1 \), we get

\[ M(z,z_1,kt) = M(Az,Bz_1,kt) \geq \emptyset \left[ \min (M(Sz,Tz_1,t), \{ M(Sz,Az,t) , M(Bz,Tz_1,t) \}, \frac{1}{2} (M(Az,Tz_1,t) + M(Bz,Az,t)) \right] \]

\[ \geq \emptyset \left[ \min (M(z,z_1,t), \{ M(z,z_1,t) \}, \frac{1}{2} (M(z_1,z_1,t) + \frac{1}{2} (M(z_1,z_1,t) + M(z_1,z_1,t)) \right] \]

\[ \geq \emptyset \left[ \min (M(z,z_1,t), \{ 1 \}, M(z_1,t) \right] \]

\[ M(z,z_1,t) \geq M(z,z_1,t) \]

Hence, from Lemma 2.2, we get \( z=z_1 \)

Thus \( z \) is the unique common fixed point of \( A, B, S \) and \( T \).

**Corollary 3.2** Let \( (X,M,*) \) be complete fuzzy metric space. Suppose that the mapping \( A,B,S \) and \( T \) are self maps of \( X \) satisfying (i-ii) conditions and there exist \( k \in (0,1) \) such that

\[ M(Ax,By,kt) \geq M(Sx,Ty,t), M(Ax,Sx,t), M(By,Ty,t), M(By,Sx,2t), M(Ax,Ty,t) \]

For every \( x,y \in X \), \( t>0 \). Then \( A,B,S \) and \( T \) have a unique common fixed point in \( X \).

**Corollary 3.3** Let \( (X,M,*) \) be complete fuzzy metric space. Suppose that the mapping \( A,B,S \) and \( T \) are self maps of \( X \) satisfying (i-ii) conditions and there exist \( k \in (0,1) \) such that

\[ M(Ax,By,kt) \geq M(Sx,Ty,t), M(Sx,Ax,t), M(Ax,Ty,t) \]

For every \( x,y \in X \), \( t>0 \). Then \( A,B,S \) and \( T \) have a unique common fixed point in \( X \).

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