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# Decomposition formulas for $H_B$ - hypergeometric functions of three variables

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***ABSTRACT:*** *In this paper we investigate several decomposition formulas associated with hypergeometric functions $H_B$ in three variables. Many operator identities involving these pairs of symbolic operators are first constructed for this purpose. By means of these operator identities, as many as 5 decomposition formulas are then found, which express the aforementioned triple hypergeometric functions in terms of such simpler functions as the products of the Gauss and Appell's hypergeometric functions.*

***Keywords:*** *Decomposition formulas; Srivastava's hypergeometric functions; Multiple hypergeometric functions; Gauss hypergeometric function; Appell's hypergeometric functions; Generalized hypergeometric function.*

## I. Introduction

Hardly there is a necessity to speak about importance of properties of hypergeometric functions for any scientist and the engineer dealing with practical application of differential equations. The solution of various problems concerning a thermal conduction and dynamics, electromagnetic oscillations and aerodynamics, a quantum mechanics and the theory of potentials, leads to hypergeometric functions. More often they appear at solving of partial differential equations by the method of a separation of variables.
A variety of the problems leading hypergeometric functions, has called fast growth of number of the functions, applied in applications (for example, in the monographs [22] 205 hypergeometric functions are studied). There were monographs and papers on the theory of special functions. But in these mono-graphs there is no formula of expansion and an analytic continuation of the generalized hypergeometric function. In this paper, using similar symbolical method Burchnall and Chaundy, we shall construct formulas of expansion for the generalized hypergeometric function. By means of the obtained formulas of expansion we find the formulas of an analytic continuation of hypergeometric function of Clausen. The found formulas of an analytic continuation express known hypergeometric Appell function.

A great interest in the theory of multiple hypergeometric functions (that is, hypergeometric functions of several variables) is motivated essentially by the fact that the solutions of many applied problems involving (for example) partial differential equations are obtainable with the help of such hypergeometric functions (see, for details, [22]; see also the recent works [12,13] and the references cited therein).
For instance, the energy absorbed by some non ferromagnetic conductor sphere included in an internal magnetic field can be calculated with the help of such functions [10]. Hypergeometric functions of several variables are used in physical and quantum chemical applications as well (cf. [11,20]).

We note that Riemann's functions and the fundamental solutions of the degenerate second- order partial differential equations are expressible by means of hypergeometric functions of several variables [6]. In investigation of the boundary value problems for these partial differential equations, we need decompositions for hypergeometric functions of several variables in terms of simpler hypergeometric functions of (for example) the Gauss and Appell's types.
Suppose that a hypergeometric function in the form (c.f. [4] and [14]) :

(1.1) $$ {}_2F_1(\alpha,\beta;\gamma;z) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} z^n $$

for $\gamma$ neither zero nor a negative integer.

*Now we* consider $H_B$ - hypergeometric function defined in [19] as follows :

(1.2) $$ H_B = H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3) $$

$$ = \sum_{n_1,n_2,n_3=0} \frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}} (-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3} $$

since

$$r=|z_1|;\ s=|z_2|;\ t=|z_3|;\ r+s+t+2\sqrt{rst}<1$$

which were introduced and investigated, over four decades ago, by Srivastava (see, for details, [16,17]; see also [19] and [20]).

Also, we study the $H_B$ - hypergeometric function, where it is regular in the unit hypersphere (c.f. [2,3]), for the $H_B$ - function, we can define as contiguous to it each of the following functions, which are samples by uppering or lowering one of the parameters by unity.

(1.3) $$H_B(\alpha+)=\sum_{n_1,n_2,n_3=0}\frac{\alpha_1+n_1+n_3}{\alpha_1}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}$$

(1.4) $$H_B(\alpha-)=\sum_{n_1,n_2,n_3=0}\frac{\alpha_1}{\alpha_1-1+n_1+n_3}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}$$

(1.5) $H_B(\alpha+,\beta+)$

$$=\sum_{n_1,n_2,n_3=0}\frac{\alpha_1+n_1+n_3}{\alpha_1}\frac{\alpha_2+n_1+n_2}{\alpha_2}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3},$$

(1.6) $$D=\sum_{j=1}^{3}d_j\ ,\ d_j=z_j\frac{\partial}{\partial z_j}\qquad;\quad j=1,2,3$$

and the way we effect it with the recursions relations as it is found in the second part of the research.

By applying the operator D in (1.6) to (1.2), we find the following set of operator identities involving the Gauss function ${}_2F_1$, the Appell's functions, and Srivastava's hypergeometric functions $H_B$ defined by (1.2) is

(1.7) $$\mathrm{D}H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3)$$

$$=\mathrm{D}\sum_{n_1,n_2,n_3=0}\frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}$$

$$=\sum_{n_1,n_2,n_3=0}\frac{(n_1+n_2+n_3)(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}$$

$$DH_B=-\frac{\alpha_1\alpha_2z_1}{\gamma_1}H_B(\alpha_1+1,\alpha_2+1,\alpha_3;\gamma_1+1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3)$$

$$-\frac{\alpha_2\alpha_3z_2}{\gamma_2}H_B(\alpha_1,\alpha_2+1,\alpha_3+1;\gamma_1,\gamma_2+1,\gamma_3;-z_1,-z_2,-z_3)$$

$$-\frac{\alpha_1\alpha_3z_3}{\gamma_3}H_B(\alpha_1+1,\alpha_2,\alpha_3+1;\gamma_1,\gamma_2,\gamma_3+1;-z_1,-z_2,-z_3)\ .$$

and

$$DH_B=-\frac{\alpha_1\alpha_2z_1}{\gamma_1}H_B(\alpha_1+,\alpha_2+;\gamma_1+)-\frac{\alpha_2\alpha_3z_2}{\gamma_2}H_B(\alpha_2+,\alpha_3+;\gamma_2+)$$

$$-\frac{\alpha_1\alpha_3z_3}{\gamma_3}H_B(\alpha_1+,\alpha_3+;\gamma_3+)$$

i.e. the partial differential equation

$$\left[DH_B + \frac{\alpha_1\alpha_2 z_1}{\gamma_1} H_B(\alpha_1+,\alpha_2+;\gamma_1+) + \frac{\alpha_2\alpha_3 z_2}{\gamma_2} H_B(\alpha_2+,\alpha_3+;\gamma_2+) + \frac{\alpha_1\alpha_3 z_3}{\gamma_3} H_B(\alpha_1+,\alpha_3+;\gamma_3+)\right] = 0$$

has a solution in the form :

$$H_B\left(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3\right) = \sum_{n_1,n_2,n_3=0} \frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}$$

## II. The main pairs of symbolic operators

Over six decades ago, Burchnall and Chaundy [1,2] and Chaundy [3] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the following inverse pairs of symbolic operators:

(2.1) $$\nabla_{z_1z_2}(h) = \frac{\Gamma(h)\Gamma(d_1+d_2+h)}{\Gamma(d_1+h)\Gamma(d_2+h)} = \sum_{k=0}^{\infty} \frac{(-d_1)_k(-d_2)_k}{(h)_k\, k!}$$

(2.2) $$\Delta_{z_1z_2}(h) = \frac{\Gamma(d_1+h)\Gamma(d_2+h)}{\Gamma(h)\Gamma(d_1+d_2+h)} = \sum_{k=0}^{\infty} \frac{(-d_1)_k(-d_2)_k}{(1-h-d_1-d_2)_k\, k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (h)_{2k}(-d_1)_k(-d_2)_k}{(h+k-1)_k(d_1+h)_k(d_2+h)_k\, k!}$$

and

(2.3) $$\nabla_{z_1z_2}(h)\Delta_{z_1z_2}(g) = \frac{\Gamma(h)\Gamma(d_1+d_2+h)\Gamma(d_1+g)\Gamma(d_2+g)}{\Gamma(d_1+h)\Gamma(d_2+h)\Gamma(g)\Gamma(d_1+d_2+g)}$$

$$= \sum_{k=0}^{\infty} \frac{(g-h)_k(g)_{2k}(-d_1)_k(-d_2)_k}{(g+k-1)_k(d_1+g)_k(d_2+g)_k\, k!}$$

$$= \sum_{k=0}^{\infty} \frac{(g-h)_k(-d_1)_k(-d_2)_k}{(h)_k(1-g-d_1-d_2)_k\, k!}$$

since

$$d_j = z_j\frac{\partial}{\partial z_j} \quad ; \qquad j=1,2.$$

We now recall here the following multivariable analogues of the Burchnall–Chaundy symbolic operators $\nabla_{z_1z_2}(h)$ and $\Delta_{z_1z_2}(h)$ defined by (2.1) and (2.2), respectively (cf. [6] and [16]; see also [18] for the case when $r = 3$):

(2.4) $$\nabla_{z_1:z_2z_3}(h) = \frac{\Gamma(h)\Gamma(d_1+d_2+d_3+h)}{\Gamma(d_1+h)\Gamma(d_2+d_3+h)} = \sum_{n_2,n_3=0}^{\infty} \frac{(-d_1)_{n_2+n_3}(-d_2)_{n_2}(-d_3)_{n_3}}{(h)_{n_2+n_3}\, n_2!n_3!}$$

since

$$d_j = z_j\frac{\partial}{\partial z_j} \quad ; \qquad j=1,2,3$$

and

(2.5) $$\Delta_{z_1:z_2z_3}(h) = \frac{\Gamma(d_1+h)\Gamma(d_2+d_3+h)}{\Gamma(h)\Gamma(d_1+d_2+d_3+h)}$$

$$= \sum_{n_2,n_3=0}^{\infty} \frac{(-d_1)_{n_2+n_3}(-d_2)_{n_2}(-d_3)_{n_3}}{(1-h-d_1-d_2-d_3)_{n_2+n_3}\, n_2!n_3!}$$

$$= \sum_{n_2,n_3=0}^{\infty} \frac{(-1)^{n_2+n_3}}{n_2!n_3!} \frac{(h)_{2(n_2+n_3)}(-d_2)_{n_2}(-d_3)_{n_3}}{(h+n_2+n_3-1)_{n_2+n_3}\, n_2!n_3!} \frac{(-d_1)_{n_2+n_3}(-d_2)_{n_2}(-d_3)_{n_3}}{(d_1+h)_{n_2+n_3}(d_2+d_3+h)_{n_2+n_3}}$$

since

$$d_j = z_j \frac{\partial}{\partial z_j} \quad ; \qquad \text{j=1,2,3.}$$

where we have applied such known multiple hypergeometric summation formulas as (cf. [9]; see also [1]

$$H_B = H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3)$$

$$= \sum_{n_1,n_2,n_3=0} \frac{(\alpha_1)_{n_1+n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{n_1!n_2!n_3!(\gamma_1)_{n_1}(\gamma_2)_{n_2}(\gamma_3)_{n_3}} (-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}$$

**since**

$$\left(\max\{|z_1|,|z_2|,|z_3|\} < 1\right)$$

## III. A set of operator identities for $H_B$ - hypergeometric functions.

By applying the pairs of symbolic operators in (2.1) to (2.5), we find the following set of operator identities involving the Gauss function ${}_2F_1$ the Appell's functions $F_1, F_2, F_3, F_4$ and $H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3)$ defined by (1.2) :

(3.1) $$H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma,\gamma;-z_1,-z_2,-z_3)$$
$$=\nabla_{z_1z_3}(\alpha_1)\nabla_{z_1z_2}(\alpha_2)\nabla_{z_2z_3}(\gamma)\,{}_2F_1(\alpha_1,\alpha_2;\gamma_1;-z_1)F_1(\alpha_3,\alpha_2,\alpha_1;\gamma;-z_2,-z_3).$$

(3.2) $$H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3)$$
$$=\nabla_{z_1z_3}(\alpha_1)\nabla_{z_1z_2}(\alpha_2)\,{}_2F_1(\alpha_1,\alpha_2;\gamma_1;-z_1)F_2(\alpha_3,\alpha_2,\alpha_1;\gamma_2,\gamma_3;-z_2,-z_3).$$

(3.3) $$H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma,\gamma;-z_1,-z_2,-z_3)$$
$$=\nabla_{z_1z_3}(\alpha_1)\nabla_{z_1z_2}(\alpha_2)\nabla_{z_2z_3}(\alpha_3)\nabla_{z_2z_3}(\gamma)\,{}_2F_1(\alpha_1,\alpha_2;\gamma_1;-z_1)$$
$$.F_3(\alpha_3,\alpha_1,\alpha_2,\alpha_3;\gamma;-z_2,-z_3).$$

(3.4) $$H_B(\alpha_1,\alpha_1,\alpha_3;\gamma_1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3)$$
$$=\nabla_{z_1z_3}(\alpha_1)\nabla_{z_1z_2}(\alpha_1)\nabla_{z_2z_3}(\alpha_1)\,{}_2F_1(\alpha_1,\alpha_1;\gamma_1;-z_1)F_4(\alpha_1,\alpha_3;\gamma_2,\gamma_3;-z_2,-z_3).$$

(3.5) $$H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3)$$
$$=\nabla_{z_1z_3}(\alpha_1)\nabla_{z_1z_2}(\alpha_2)\nabla_{z_2z_3}(\alpha_3)\,{}_2F_1(\alpha_1,\alpha_2;\gamma_1;-z_1)$$
$$.\,{}_2F_1(\alpha_2,\alpha_3;\gamma_2;-z_2)\,{}_2F_1(\alpha_1,\alpha_3;\gamma_3;-z_3).$$

In view of the known Mellin–Barnes contour integral representations for the Gauss function ${}_2F_1$, the Appell's functions $F_1, F_2, F_3, F_4$, and Srivastava's triple hypergeometric functions $H_B$, it is not difficult to give alternative proofs of the operator identities (3.1) to (3.5) above by using the Mellin and the inverse Mellin transformations (see, [20,22]). The details involved in these alternative derivations of the operator identities (3.1) to (3.5) are being omitted here.

## IV. Decompositions for $H_B$ - hypergeometric functions.

Making use of the principle of superposition of operators, from the operator identities (3.1) to (3.5) we can derive the following decomposition formulas for $H_B$ - hypergeometric functions:

(4.1) $$H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma,\gamma;-z_1,-z_2,-z_3)=$$

$$\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_1)_{n_1+n_2+n_3}(\alpha_2)_{n_1+n_2+n_3}(\alpha_3)_{2n_1+n_2+n_3}}{(\gamma_1)_{n_2+n_3}(\gamma)_{n_1}(\gamma)_{2n_1+n_2+n_3}}\frac{(-z_1)^{n_2+n_3}(-z_2)^{n_1+n_2}(-z_3)^{n_1+n_3}}{n_1!n_2!n_3!}$$
$$\cdot\,{}_2F_1(\alpha_1+n_1+n_2+n_3,\alpha_2+n_1+n_2+n_3;\gamma_1+n_2+n_3;-z_1)$$

$$.\,F_1(\alpha_3+2n_1+n_2+n_3,\alpha_2+n_1+n_2,\alpha_1+n_1+n_2+n_3;\gamma+2n_1+n_2+n_3;-z_2,-z_3);$$

(4.2) $$H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3)=$$

$$\sum_{n_1,n_2=0}^{\infty}\frac{(\alpha_1)_{n_1+n_2}(\alpha_2)_{n_1+n_2}(\alpha_3)_{n_1+n_2}}{(\gamma_1)_{n_1+n_2}(\gamma_2)_{n_2}(\gamma_3)_{n_1}}\frac{(-z_1)^{n_1+n_2}(-z_2)^{n_2}(-z_3)^{n_1}}{n_1!n_2!}$$

$$\cdot\,{}_2F_1(\alpha_1+n_1+n_2,\alpha_2+n_1+n_2;\gamma_1+n_1+n_2;-z_1)$$

$$.\,F_2(\alpha_3+n_1+n_2,\alpha_2+n_1+n_2,\alpha_1+n_1;\gamma_2+n_2,\gamma_3+n_1;-z_2,-z_3);$$

(4.3) $$H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma,\gamma;-z_1,-z_2,-z_3)=$$

$$\sum_{n_1,n_2,n_3,n_4=0}^{\infty}\frac{(\alpha_1)_{n_2+2n_3+n_4}(\alpha_2)_{n_1+n_2+n_3+n_4}(\alpha_3)_{2n_1+n_2+n_3}(\alpha_3)_{2n_1+n_2+n_4}}{(\alpha_3)_{2n_1+n_2}(\gamma)_{n_1}(\gamma)_{2n_1+2n_2+n_3+n_4}(\gamma_1)_{n_3+n_4}}\frac{(-z_1)^{n_3+n_4}(-z_2)^{n_1+n_2+n_3}(-z_3)^{n_1+n_2+n_4}}{n_1!n_2!n_3!n_4!}$$

$$\cdot\,{}_2F_1(\alpha_1+n_2+2n_3+n_4,\alpha_2+n_1+n_2+n_3+n_4;\gamma_1+n_3+n_4;-z_1)$$
$$.\,F_3\left(\begin{array}{l}\alpha_3+2n_1+n_2+n_3,\alpha_1+n_1+n_2+n_3+n_4,\alpha_2+n_1+n_2+n_3,\alpha_3+2n_1+n_2+n_4;\\ \\ \gamma+2n_1+2n_2+n_3+n_4;-z_2,-z_3\end{array}\right);$$

(4.4) $$H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3)=$$

$$\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_1)_{n_1+n_2+n_3}(\alpha_2)_{n_1+n_2+n_3}(\alpha_3)_{n_1+n_2}(\alpha_3)_{n_2+n_3}}{(\alpha_3)_{n_2}(\gamma_1)_{n_1+n_3}(\gamma_2)_{n_1+n_2}(\gamma_3)_{n_2+n_3}}\frac{(-z_1)^{n_1+n_3}(-z_2)^{n_1+n_2}(-z_3)^{n_2+n_3}}{n_1!n_2!n_3!}$$

$$\cdot\,{}_2F_1(\alpha_1+n_1+n_2+n_3,\alpha_2+n_1+n_2+n_3;\gamma_1+n_1+n_3;-z_1)$$

$$\cdot\,{}_2F_1(\alpha_2+n_1+n_2,\alpha_3+n_1+n_2;\gamma_2+n_1+n_2;-z_2)$$

$$\cdot\,{}_2F_1(\alpha_3+n_2+n_3,\alpha_1+n_1+n_2+n_3;\gamma_3+n_2+n_3;-z_3);$$

(4.5) $H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma,\gamma;-z_1,-z_2,-z_2)=$

$$\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\alpha_1)_{n_1+n_2+n_3}(\alpha_2)_{n_1+n_2+n_3}(\alpha_3)_{2n_1+n_2+n_3}}{(\gamma_1)_{n_2+n_3}(\gamma)_{n_1}(\gamma)_{2n_1+n_2+n_3}}\frac{(-z_1)^{n_2+n_3}(-z_2)^{2n_1+n_2+n_3}}{n_1!n_2!n_3!}$$

$$\cdot\,{}_2F_1(\alpha_1+n_1+n_2+n_3,\alpha_2+n_1+n_2+n_3;\gamma_1+n_2+n_3;-z_1)$$

$$\cdot\,{}_2F_1(\alpha_3+2n_1+n_2+n_3,\alpha_1+\alpha_2+2n_1+2n_2+n_3;\gamma+2n_1+n_2+n_3;-z_2).$$

Decomposition (3.3) can be proved by means of equality

(4.6) $$\nabla_{z_1z_3}(\alpha_1)\nabla_{z_1z_2}(\alpha_2)\nabla_{z_2z_3}(\alpha_3)\nabla_{z_2z_3}(\gamma)=\frac{1}{(\alpha_1)_m(\alpha_1)_p(\alpha_2)_m(\alpha_3)_n(\alpha_3)_p}$$

$$\cdot\sum_{n_1,n_2,n_3,n_4=0}^{\infty}\frac{(\alpha_1)_{n_2+n_3}(\alpha_1)_{n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)^2_{n_1}(\alpha_1+n_2+n_3)_m(\alpha_1+n_3)_p}{(\alpha_1)_{n_1+n_2+n_3+n_4}(\alpha_2)_{n_1+n_2+n_3}}$$

$$\cdot\frac{(\alpha_2+n_1+n_2)_m(\alpha_3+n_1)_n(\alpha_3+n_1)_p}{(\alpha_3)_{2n_1+n_2}(\gamma)_{n_1}}\frac{(-d_1)_{n_3+n_4}(-d_2)_{n_1+n_2+n_3}(-d_3)_{n_1+n_2+n_4}}{n_1!n_2!n_3!n_4!}.$$

Taking into account the identities (4.6), from parity (3.3), we have

(4.7) $H_B(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma,\gamma;-z_1,-z_2,-z_3)$

$$\sum_{n_1,n_2,n_3,n_4=0}^{\infty}\frac{(\alpha_1)_{n_2+n_3}(\alpha_1)_{n_3}(\alpha_2)_{n_1+n_2}(\alpha_3)^2_{n_1}}{(\alpha_1)_{n_1+n_2+n_3+n_4}(\alpha_2)_{n_1+n_2+n_3}(\alpha_3)_{2n_1+n_2}(\gamma)_{n_1}n_1!n_2!n_3!n_4!}$$

$$\cdot\;(-d_1)_{n_3+n_4}\,F\,(\alpha_1+n_2+n_3,\alpha_2+n_1+n_2;\gamma_1;(-z_1))$$

$$\cdot\;(-d_2)_{n_1+n_2+n_3}(-d_3)_{n_1+n_2+n_4}\,F_3(\alpha_3+n_1,\alpha_1+n_3,\alpha_2,\alpha_3+n_1;\gamma;(-z_2),(-z_3));$$

we have

(4.8) $$(-d_1)_{n_3+n_4}\,F\,(\alpha_1+n_2+n_3,\alpha_2+n_1+n_2;\gamma_1;(-z_1))$$

$$=(-1)^{n_3+n_4}(-z_1)^{n_3+n_4}\frac{(\alpha_1)_{n_2+2n_3+n_4}(\alpha_2)_{n_1+n_2+n_3+n_4}(\alpha_3)^2_{n_1}}{(\alpha_1)_{n_2+n_3}(\alpha_2)_{n_1+n_2}(\gamma_1)_{n_3+n_4}}$$

$$\cdot\;F\,(\alpha_1+n_2+2n_3+n_4,\alpha_2+n_1+n_2+n_3+n_4;\gamma_1+n_3+n_4;(-z_1));$$

(4.9) $$(-d_2)_{n_1+n_2+n_3}(-d_3)_{n_1+n_2+n_4}\,F_3(\alpha_3+n_1,\alpha_1+n_3,\alpha_2,\alpha_3+n_1;\gamma;(-z_2),(-z_3))$$

$$=(-1)^{n_3+n_4}(-z_2)^{n_1+n_2+n_3}(-z_3)^{n_1+n_2+n_4}$$

$$\cdot\frac{(\alpha_1)_{n_1+n_2+n_3+n_4}(\alpha_2)_{n_1+n_2+n_3}(\alpha_3)_{2n_1+n_2+n_3}(\alpha_3)_{2n_1+n_2+n_4}}{(\alpha_1)_{n_3}(\alpha_3)^2_{n_1}(\gamma)_{2n_1+2n_2+n_3+n_4}}$$

$$\cdot\,F_3\left(\begin{matrix}\alpha_3+2n_1+n_2+n_3,\alpha_1+n_1+n_2+n_3+n_4,\alpha_2+n_1+n_2+n_3,\\ \\ \alpha_3+2n_1+n_2+n_4;\gamma+2n_1+2n_2+n_3+n_4;(-z_2),(-z_3)\end{matrix}\right).$$

Substituting identities (4.8) - (4.9) into equality (4.7), we get

$$H_B\left(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma,\gamma;-z_1,-z_2,-z_3\right)$$
$$=\nabla_{z_1z_3}\left(\alpha_1\right)\nabla_{z_1z_2}\left(\alpha_2\right)\nabla_{z_2z_3}\left(\alpha_3\right)\nabla_{z_2z_3}\left(\gamma\right)\,{}_2F_1\left(\alpha_1,\alpha_2;\gamma_1;-z_1\right)$$
$$.\,F_3\left(\alpha_3,\alpha_1,\alpha_2,\alpha_3;\gamma;-z_2,-z_3\right).$$

Our operational derivations of the decomposition formulas (4.1) to (4.5) would indeed run parallel to those presented in the earlier works which we have already cited in the preceding sections.

## V. Integral representations via decomposition formulas

Next we turn to a set of known double-integral representations of the Laplace type for $H_B$, each of which was derived by Srivastava [20] from the following rather elementary formula :

(5.1) $$(\lambda)_n=\frac{1}{\Gamma(\lambda)}\int_0^{\infty}e^{-t}t^{\lambda+n-1}dt$$

since $R(\lambda)>0;\quad n\in N_0.$

For each of the hypergeometric functions $H_B$. Srivastava [19,20] gave several ordinary as well as contour integral representations of the Eulerian, Laplace, Mellin– Barnes, and Pochhammer's double-loop types. Here, in this section, we first observe that several known integral representations of the Eulerian type can be deduced also from the corresponding decomposition formulas of Section 4. For example, we have [19] .

(5.2) $$H_B\left(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3\right)$$
$$=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\infty}\int_0^{\infty}e^{-s-t}t^{\alpha_1-1}s^{\alpha_2-1}\,{}_0F_1\left(-;\gamma_1;-z_1s\,t\right)\Psi_2\left(\alpha_3;\gamma_2,\gamma_3;-z_2s,-z_3t\right)ds\,dt$$

since

$$\min\left\{R\left(\alpha_1\right),R\left(\alpha_2\right)\right\}>0\quad;\quad\max\left\{R\left(z_2\right),R\left(z_3\right)\right\}<1\ ,$$

where $\Psi_2$ denotes one of Humbert's confluent hypergeometric functions of two variables :

(5.3) $$\Psi_2\left(\alpha_1;\gamma_1,\gamma_2;-z_1,-z_2\right)=\sum_{n_1,n_2=0}^{\infty}\frac{(\alpha_1)_{n_1+n_2}}{(\gamma_1)_{n_1}(\gamma_2)_{n_2}}\frac{\left(-z_1\right)^{n_1}\left(-z_2\right)^{n_2}}{n_1!\,n_2!}$$

and

(5.4) $$\Psi_2\left(\alpha_1;\gamma_1,\gamma_2;-z_1,-z_2\right)=\int_0^{\infty}e^{-t}t^{\alpha_1-1}\,{}_0F_1\left(-;\gamma_1;-z_1t\right){}_0F_1\left(-;\gamma_1;-z_2t\right)dt$$

since

$$R\left(\alpha_1\right)>0\ ,$$

which is easily derivable by combining (5.1) with the definition (5.3), we find from Srivastava's result (5.2) that

(5.5) $$H_B\left(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\gamma_2,\gamma_3;-z_1,-z_2,-z_3\right)$$

$$=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-s-t-u}t^{\alpha_1-1}s^{\alpha_2-1}u^{\alpha_3-1}$$

$$.\,{}_0F_1\left(-;\gamma_1;-z_1s\,t\right){}_0F_1\left(-;\gamma_2;-z_2u\,s\right){}_0F_1\left(-;\gamma_3;-z_3u\,t\right)ds\,dt\,du$$

since

$$\min\left\{R\left(\alpha_1\right),R\left(\alpha_2\right),R\left(\alpha_3\right)\right\}>0\ .$$

## VI. Concluding remarks and observations

By suitably specializing the decomposition formulas (4.1) to (4.5), we can deduce a number of (known or new) decomposition formulas including those given by (for example) Burchnall and Chaundy [1,2]. For instance, for Appell's hypergeometric functions, we find the following (presumably new) results:

$$(6.1)\qquad F_1\left(\alpha,\beta;\gamma;-z_1,-z_2\right)=\sum_{i,j=0}^{\infty}\frac{(\alpha)_{2i+2j}(\beta)_{i+j}(\gamma)_{2i}}{(\gamma+i-1)_i\left[(\gamma)_{2i+j}\right]^2}\frac{\left(-z_1\right)^{i+j}\left(-z_2\right)^{i+j}}{i!\,j!}$$

$$.\,F_4\left(\alpha+2i+2j,\beta+i+j;\gamma+2i+j,\gamma+2i+j;-z_1,-z_2\right)$$

and

$$(6.2)\qquad F_1\left(\alpha_1,\alpha_2,\alpha_3;\gamma,\gamma;-z_1,-z_2\right)=\sum_{i,j=0}^{\infty}\frac{(\alpha_1)_{2i+j}(\alpha_2)_{i+j}(\alpha_3)_{i+j}}{(\gamma)_i(\gamma)_{2i+2j}}\frac{\left(-z_1\right)^{i+j}\left(-z_2\right)^{i+j}}{i!\,j!}$$

$$.\,F_3\left(\alpha_1+2i+j,\alpha_1+2i+j,\alpha_2+i+j,\alpha_3+i+j;\gamma+2i+j;-z_1,-z_2\right).$$

Furthermore, by making use of the decompositions (4.2), we can derive the following known reduction formulas for Srivastava's triple hypergeometric function $H_B$ [19]:

$$(6.3)\qquad H_B\left(\alpha_1,\alpha_2,\alpha_3;\gamma_1,\alpha_3,\alpha_3;-z_1,-z_2,-z_3\right)$$

$$=\left(1+z_2\right)^{-\alpha_2}\left(1+z_3\right)^{-\alpha_1}F_4\left(\alpha_1,\alpha_2;\gamma_1,\alpha_3;\frac{-z_1}{\left(1+z_2\right)\left(1+z_3\right)},\frac{(-z_2)(-z_3)}{\left(1+z_2\right)\left(1+z_3\right)}\right).$$

Some of the most recent contributions in the theory of Srivastava's $H_B$ - hypergeometric series include a paper by Harold Exton [5] and a paper by Rathie and Kim [15]. The work of Exton [5] made use of elementary series manipulation and some wellknown analytic continuation formulas for the Gauss hypergeometric function in order to derive a fundamental set of nine solutions of the system of partial differential equations satisfied by the symmetrical function $H_B$, Rathie and Kim [15].

$$(6.4)\qquad F_1\left(a,b,b';a+b-b'+1;1,-1\right)=\frac{\Gamma(a+b-b'+1)\Gamma(1-b')\Gamma\left(\frac{1}{2}a+1\right)}{\Gamma(a+1)\Gamma(b-b'+1)\Gamma\left(\frac{1}{2}a-b'+1\right)}$$

since $\qquad R\left(b'\right)<1$.

However, in deriving many of their applications of the summation formula (6.4), Rathie and Kim [15] obviously violated the constraint $R\left(b'\right)<1$ associated with (6.4) at least in situations in which the hypergeometric series involved in their investigation would not terminate.

Finally, we note that, here in this paper, we have not applied such superpositions of operators as those provided by (for example)

$$\nabla_{z_1z_2}(\alpha)\nabla_{z_1z_3}(\alpha)\Delta_{z_2z_3}(\alpha)\Delta_{z_2z_3}(\gamma)\quad;\quad\nabla_{z_1z_3}(\alpha)\nabla_{z_1z_2}(\alpha)\Delta_{z_2z_3}(\alpha)$$

and

$$\nabla_{z_1z_3}(\alpha)\nabla_{z_1z_2}(\alpha)\Delta_{z_2z_3}(\alpha)\Delta_{z_2z_3}(\gamma)\tilde{\Delta}_{z_1;z_2z_3}(\gamma)$$

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