

Simulation of Simple Pendulum

Abdalfthah Elbori¹, Ltfei Abdalsmd²

¹(MODES Department Atilim University Turkey)

²(Material Science Department, Kastamonu University Turkey)

Abstract : This paper presents the Physics Rotational Method of the simple gravity pendulum, and it also applies Physics Direct Method to represent these equations, in addition to the numerical solutions discusses. This research investigates the relationship between angular acceleration and angle to find out different numerical solution by using simulation to see their behavior which shows in last part of this article.

Keywords: Physics Rotational Method, Simple Pendulum, Numerical Solution, Simulation

I. INTRODUCTION

A pendulum is a weight suspended from a pivot so that it can swing freely. When a pendulum is displaced sideways from its resting equilibrium position, it is subject to a restoring force due to gravity that will accelerate it back toward the equilibrium position [1] and [2]. When released, the restoring force combined with the pendulum's mass causes it to oscillate about the equilibrium position, swinging back and forth. The time for one complete cycle, a left swing and a right swing, is called the period. The period depends on the length of the pendulum and also on the amplitude of the oscillation. However, if the amplitude is small, the period is almost independent of the amplitude [3] and [4].

Many things in nature wiggle in a periodic fashion. That is, they vibrate. One such example is a simple pendulum. If we suspend a mass at the end of a piece of string, we have a simple pendulum. Here, we try to represents a periodic motion used in times past to control the motion of grandfather and cuckoo clocks. Such oscillatory motion is called simple harmonic motion[5]. It was Galileo who first observed that the time a pendulum takes to swing back and forth through small distances depends only on the length of the pendulum[3]. In this paper applies Physics - Rotational Method and Physics Direct Method to represent these equations. In the second step, it discusses Numerical Solution and Linearization of simple pendulum. In the third step in this paper figures out how to use Simulation of Simple pendulum and compare with double pendulum, finally we will find the relationship between angular acceleration and angle to investigate the numerical solution and simulation to see their behavior.

II. PHYSICS – ROTATIONAL METHOD

The pendulum is modeled as a point mass at the end of a massless rod. We define the following variables as in the figure 1:

θ = angle of pendulum (0=vertical).

R = length of rod.

T = tension in rod.

m = mass of pendulum.

g = gravitational constant.

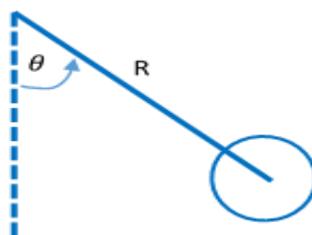


Figure.1. Pendulum variables

We will derive the equation of motion for the pendulum using the rotational analog of Newton's second law for motion about a fixed axis, which is $\tau = I \alpha$, where τ = net torque, I = rotational inertia $\alpha = \theta''$ = acceleration. The rotational inertia about the pivot is $I = mR^2$. Torque can be calculated as the vector cross product of the position vector and the force. The magnitude of the torque due to gravity works out to be $\tau = -Rmg \sin \theta$. So, we have:

$$-Rmg \sin \theta = mR^2 \alpha$$

Which simplifies to

$$\theta'' = -\frac{g}{R} \sin(\theta) \quad (1)$$

This is the equation of motion for the pendulum.

III. PHYSICS - DIRECT METHOD

Let us consider Newton's second law $\vec{F} = m\vec{a}$. To show that there is nothing new in the rotational version of Newton's second law, we derive the equation of motion here without the rotational dynamics. As you will see, this method involves more algebra. We'll need the standard unit vectors, \vec{i}, \vec{j} as in the Figure 2.

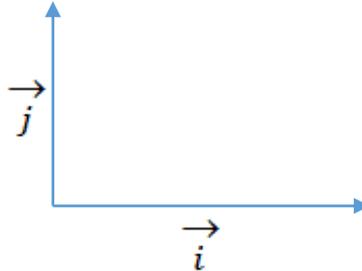


Figure.2. standard unit vectors

\vec{i} = unit vector in horizontal direction, \vec{j} = unit vector in vertical direction.

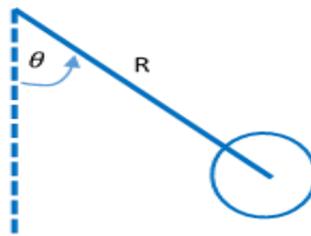


Figure.3. One Pendulum

By Figure 3 is easy to derive the kinematics of the pendulum as follows:

$$\text{position} = R\sin\theta\vec{i} - R\cos\theta\vec{j},$$

$$\text{velocity} = R\theta'\cos\theta\vec{i} + R\theta'\sin\theta\vec{j},$$

$$\text{acceleration} = R(\theta''\cos\theta\vec{i} - \theta'^2\sin\theta\vec{i} + \theta''\sin\theta\vec{j} + \theta'^2\cos\theta\vec{j}).$$

The position is derived by a fairly simple application of trigonometry. The velocity and acceleration are then the first and second derivatives of the position. Next, we draw the free body diagram for the pendulum. The forces on the pendulum are the tension in the rod T and gravity. So, it is possible to write the net force as in the figure 4:

$$\vec{F} = T\cos\theta\vec{j} - T\sin\theta\vec{i} - mg\vec{j}$$

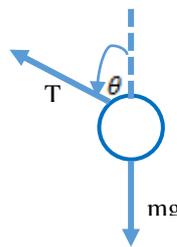


Figure.4. Pendulum forces

Using Newton's law $\vec{F} = m\vec{a}$ and the pendulum acceleration we found earlier, we have

$$T\cos\theta\vec{j} - T\sin\theta\vec{i} - mg\vec{j} = mR(\theta''\cos\theta\vec{i} - \theta'^2\sin\theta\vec{i} + \theta''\sin\theta\vec{j} + \theta'^2\cos\theta\vec{j})$$

It is possible to rewrite the vector components of the above equation as separate equations. This gives us two simultaneous equations: the first components is \vec{i} and the second is for the \vec{j} component.

$$-T\sin\theta = mR(\theta''\cos\theta - \theta'^2\sin\theta) \quad \& \quad T\cos\theta - mg = mR(\theta''\sin\theta + \theta'^2\cos\theta)$$

Next, by doing some algebraic manipulations to eliminate the unknown T . Multiply the first equation by $\cos\theta$ and the second by $\sin\theta$.

$$-T\sin\theta\cos\theta = mR(\theta''\cos^2\theta - \theta'^2\sin\theta\cos\theta) \quad (2)$$

$$T\cos\theta\sin\theta - mg\sin\theta = mR(\theta''\sin^2\theta + \theta'^2\cos\theta\sin\theta) \quad (3)$$

By adding the equation (2) and (3) together, we obtain

$$-mg\sin\theta = mR(\theta''\cos^2\theta + \theta''\sin^2\theta)$$

Therefore,

$$\theta''\cos^2\theta + \theta''\sin^2\theta = \theta'',$$

hence,

$$\theta'' + \frac{g}{R}\sin\theta = 0$$

IV. NUMERICAL SOLUTION AND LINEARIZATION

Let us discuss how we can solve the equations of motion numerically, so that we can drive the simulation, we use the Runge-Kutta method for solving sets of ordinary differential equations. Firstly let us define a variable for the angular velocity $\omega = \theta'$. Then we can write the second order equation (1) as two first order equations. Such as,

$$\omega = \theta' \Rightarrow \omega' = \frac{g}{R}\sin\theta$$

$$u_1 = \theta, u_2 = \theta' \text{ Then, } u'_1 = \theta' = u_2 \text{ \& } u'_2 = \theta'' = -\omega\sin u_1$$

$$\left. \begin{array}{l} u'_1 = u_2 \\ u'_2 = -\omega\sin u_1 \end{array} \right\} \quad (4)$$

The restoring force must be proportional to the negative of the displacement. Here we have: $F = -mg\sin\theta$, which is proportional to $\sin\theta$ and not to θ itself.

By using Taylor series to get approximation of the function $\sin\theta$ at the equilibrium point $\theta = 0$, we obtain $\sin\theta \approx \theta$.

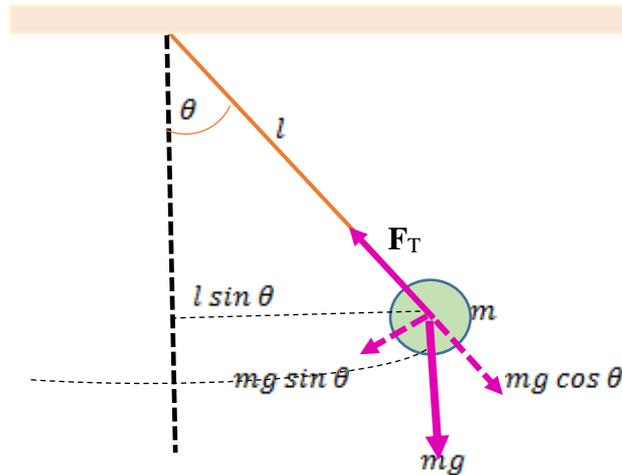


Figure.5. Motion of Simple Pendulum

$$\omega = \theta' \Rightarrow \omega' = \frac{g}{R}\sin\theta$$

Therefore, for small angles, we have

$$F \approx -\frac{mg}{l}x, \text{ where } x = l\theta.$$

We have also the period and frequency are: $T = 2\pi\sqrt{\frac{L}{g}}$, and $f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$,

$$\tau = I\alpha = fl = -mgl\sin\theta, \text{ where } I = ml^2$$

$$-mgl\sin\theta = ml^2\frac{d^2\theta}{dt^2} \Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \text{ and let } \omega^2 = \frac{g}{l}$$

We obtain

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \quad (5)$$

V. SIMULATION OF SIMPLE PENDULUM.

This simulation shows a simple pendulum operating under gravity. For small oscillations, the pendulum is linear, but it is non-linear for larger oscillations.

We have $\theta'' = -\frac{g}{R}\sin\theta = -\omega\sin\theta$ where $\omega = \frac{g}{R}$ is positive constant. Hence

$$\theta'' + \omega\sin\theta = 0 \quad (6)$$

let us use represent state to solve this ordinary differential equation (6). Let's start with a same step

$$u_1 = \theta, u_2 = \theta'.$$

Then,

$$\left. \begin{aligned} u'_1 &= \theta' = u_2 & u'_2 &= \theta'' = -\omega\sin u_1 \\ u'_1 &= u_2 \\ u'_2 &= -\omega\sin u_1 \end{aligned} \right\} \quad (7)$$

Open up a m-file and type:

```
function dudt=pendulum (t,u)
w=10;
dudt(1,:) = u(2);
dudt(2,:) = -w*sin(u(1));
end
```

we note that, by looking at the figure.6, a number of frequencies increase as we decrease the value w . Let us save this file as "pendulum m". We need to use MATLAB to solve l the differential equation by this code

```
clear all
tspan=[0,2*pi];
u0=[pi/4;0];
[t,u]=ode45(@pendulum,tspan,u0,[]);
plot(t,u(:,1),'b-', 'Line Width', 2)
xlabel('time')
ylabel('angle')
```

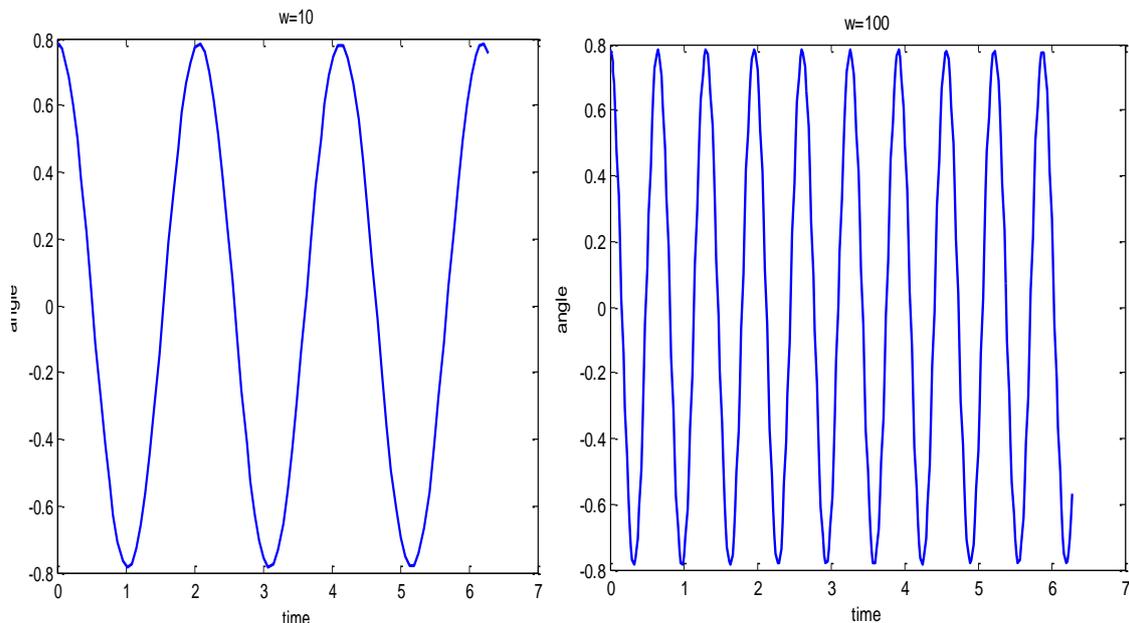


Figure.6 Angular Displacement

Note that as we increase ω then oscillations will increase. By another word if we decrease R = length of rod we will increase number of oscillations

V.1 Euler method

We can also use Euler method, let us describe here is the code for the numerical solution of the equations of motion for a simple pendulum using the Euler method. There is important this, we note that these oscillations grow with time. The MATLAB code has already existed [1], just we insert our data as in the figure.7

```
length= 1;
g=9.8;
npoints = 250;
dt = 0.04;
omega = zeros(npoints,1);
theta = zeros(npoints,1);
time = zeros(npoints,1);
theta (1)=0.2;
for step = 1: npoints-1
omega(step+1) = omega(step) - (g/length) *theta(step)*dt;
theta(step+1) = theta(step)+omega(step)*dt
time(step+1) = time(step) + dt;
end
plot (time,theta, 'r' ); xlabel('time (seconds) ');
ylabel ('theta (radians)');
```

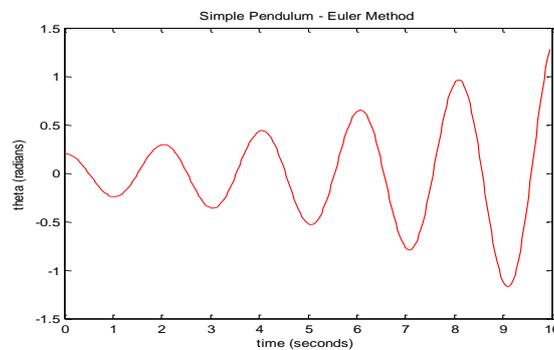


Figure.7 Numerical Solution by Euler Method

VI. SIMULATION OF SIMPLE NON-LINEAR.

Although finding an analytical solution to the simple non-linear model is not evident, it can be solved numerically with the aid of MATLAB-Simulink. Start by isolating the highest order derivative from the linear model:

$$\theta'' + \frac{g}{l} \sin\theta = 0 \Rightarrow \theta'' = -\frac{g}{l} \sin\theta$$

The MATLAB-Simulink model that solves this equation is given by the figure.8.

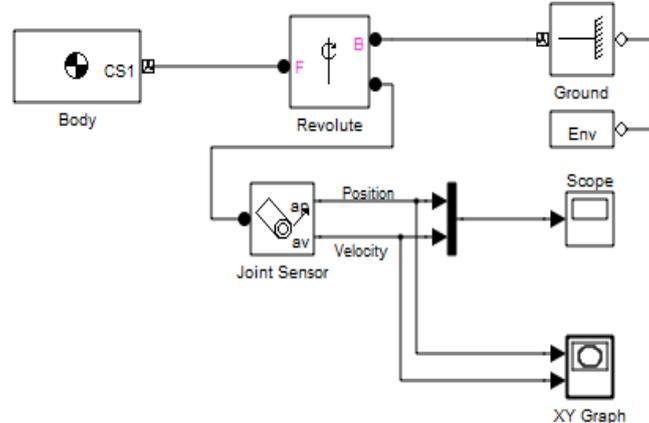


Figure.8 Simulation of Simple Nonlinear Pendulum

Note that the drag force must change sign based on the angular velocity of the pendulum. In other words, although the drag force is taken as constant, it must always act opposite to the direction of motion. If we run the figure.9, we obtain figure.9 and it depends on initial condition and the mass =55 and time=12 second, inertia= [1.25e-4 0 0; 0 0.208 0; 0 0 0.208], we obtain

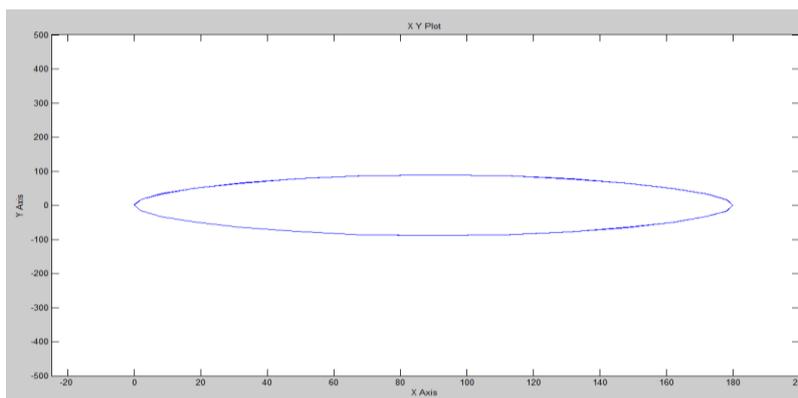


Figure.9 coordinates axes

VII. CONCLUSION

The length of the simple pendulum is directly proportional to the square of the period of oscillation, we found that the pendulum goes slower than simple pendulum theory at larger angles. The time of this motion is called the period, the period does not depend on the mass of the pendulum or on the size of the arc through which it swings. Another factor involved in the period of motion is, the acceleration due to gravity (g), which on the earth is 9.8 m/s^2 . It follows then that a long pendulum has a greater period than a shorter pendulum.

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