International Journal of Engineering Science Invention
ISSN (Online): 2319 – 6734, ISSN (Print): 2319 – 6726
www.ijesi.org ||Volume 6 Issue 5|| May 2017 || PP. 01-07

Solving Exponential Equations: Learning from the Students We Teach

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Abstract: *The purpose of this paper is to share with the mathematics community what I discovered from analyzingone of myGrade 11 students' approach to solving exponential equations of the form* $a^{x+p} \pm a^{x+q} = k$ *, where* a *and* k *are positive integers greater than 1, and* $p, q \in Z$ *. The student got a correct answer using a procedure which does not conform with the known exponential laws, thereby making it difficult to evaluate the student's work. I gave the student's script to fellow mathematics educators and they marked the student wrong, arguing that laws of exponents cannot be extended to a sum or difference of exponentials with the same base. Ithen decided to present the student's solution method to other mathematics experts (through the Math Forum)for further evaluation. The responses and comments I received were far from being conclusive. It was suggested that there was need to use mathematical proof to verify whether the student's approach was valid or not. I then set out to investigate why what looked like invalid reasoning on the surface gave the right answer in the end.After a careful analysis of the student's approach, coupled with some kind of empirical investigations and mathematical proof, I eventually discovered that there was some logic in the student's approach, only that it was not supported by the known theory of exponents. I therefore recommend that mathematics educators should not take students' solution methods for granted. We might be marking some of the students' solution methods wrong when they are valid, and robbing them of their precious marks. It is possible for students to come up with new and valid methods of solving mathematics problems which are not known to the educators.Good mathematics educators should therefore be on the lookout for new unanticipated approaches to solving mathematical problems that students of exceptional abilities may use in class. The famous German mathematician and astronomer, Carl Friedrich Gauss (1777-1855)amazed his teachers when he discovered a quick way of summing the integers from 1 to 100, at the age of seven. Such exceptional intellectual abilities still existeven in school children of today.*

Key Terms*: Exponents, exponential equations, anticipated approach, unanticipated approac*h

I. Introduction

Exponents, which indicate how many times a number multiplies itself, are fundamental in the modern technological world.Exponents are used by Computer Programmers, Bankers, Investors, Accountants, Economists, Insurance Risk Assessors, Chemists, Geologists, Biologists, Physicists, artisans, Sound Engineers and Mathematicians, to solve real life problems [1]. For instance, investors simply plug in numbers into exponential equations and they are able to figure out how much they are earning on their savings. Banks use exponential equations to calculate how much borrowers have to repay each month to settle their loans. Geoscientists use exponents to calculate the intensity of an earthquake. Builders and carpenters use exponents daily to calculate the quantity of material needed to construct buildings. Aeronautical engineers apply knowledge of exponents to predict how rockets and jets will perform during a flight.Clearly, exponents are a necessary component of the school mathematics curriculum to prepare students for these critical careers.

In South Africa, Grade 10 and 11 students are expected to use laws of exponents to solve exponential equations[2], including those of the form $a^{x+p} \pm a^{x+q} = k$. Examples of such equations are:

- $3^{x} + 3^{x+2} = 10$
- $3^{x+2} - 3^{x+1} + 3^{x+3} = 11$
- $2^{x} - 2^{x-2} = 3$
- $5^{x+1} + 5^{x} = 6$

In solving these equations, we rely mainly on the exponential theorem which states that:

If $a^{x} = a^{c}$, then $x = c$.

It is not known whether this could be extended toa sum or difference of exponentials with the same base.For instance, we do not have theorems suggesting that:

$a^{x+p} + a^{x+q} = a^{r} + a^{s} \Rightarrow x + p + x + q = r + s$

However, one of my Grade 11 students used a technique that seemed to suggest the existence of such theorems. The student got a correct answer using a procedure that seemed to be invalid because it did not conform with the known theorems of exponents. I therefore set out to investigate the reason why an approach that seemed invalid on the surface, gave a correct answer in the end. In this paper, I intend to share with the mathematics community what I discovered from analysing the student method of solution.

II. Theoretical Framework

Mathematics is "a human activity that involves observing, representing and investigating patterns and qualitative relationships in the physical and social phenomena and between mathematical objects themselves" [2, p. 8]. Mathematical problem solvingteaches us to think critically, logically and creatively [2].Contemporary views of mathematics education encourage students to use their own methods to solve problems rather than imitate their teachers. This is in line with the constructivist view of mathematics teaching and learning which states that students have the ability to construct their own knowledge through discovery and problem solving. Constructivism "gives preeminent value to the development of students' personal mathematical ideas" [3,p. 9].In a traditional mathematics classroom, it is predetermined that students will solve mathematics problems using the method(s) shown to them by their teacher. This is in sharp contrast to a constructivist mathematics classroom where students employ different and multiple solution methods. "Students may use unanticipated solution-methods and unforeseen difficulties may arise"[4, p. 3].The role of the teacher in this context becomes more demanding and unless the teacherlooks more closely at what the students have written, some solutionsrisk being marked wrong when they are valid.A good mathematics teacher understands that there are multiple ways of solving mathematics problems, and hence seriously considersevery attempt that students make towards solving a mathematics problem.

III. Purpose of The Study

The purpose of this study was to explore the logic behind one of my Grade 11 students' unanticipated approach to solving exponential equations of the form $a^{x+p} \pm a^{x+q} = k$,which resulted in a correct answer, using what seemed to be an invalid procedure. Findings of this study were intended to assist the researcher in deciding whether or not the student's method could be accepted as valid.

IV. Methodology

This investigation utilized the single-case study design with only one Grade 11 student as the unit of analysis. According to Yin [5], a single-case study is appropriate under the following circumstances: (a) where it represents a critical case that can be used to confirm, challenge or extend a well-formulated theory, (b) where it represents a unique case, and (c) where it makes a significant contribution to knowledge or theory building. All the three circumstances were found to match the present investigation perfectly.Data were collected from the student's classwork book. The focus of the study was on the student's approach to solving exponential equations of the form $a^{x+p} \pm a^{x+q} = k$. The student's solution method which was a unique case, was presented to the Math Forum for evaluation. Two mathematics experts, Doctor X and Doctor Y (not their real names), responded through email. After analyzing their comments, I decided to conduct my own empirical investigations by applying the student's unanticipated approach to several other similar exponential equations and compared the results with those obtained using the anticipated (usual) approach.The findings of the study are presented in the next section.

V. Findings

SOLVING EXPONENTIAL EQUATIONS OF THE FORM $a^{x+p} \pm a^{x+q} = k$

The following is what the student wrote in her classwork:

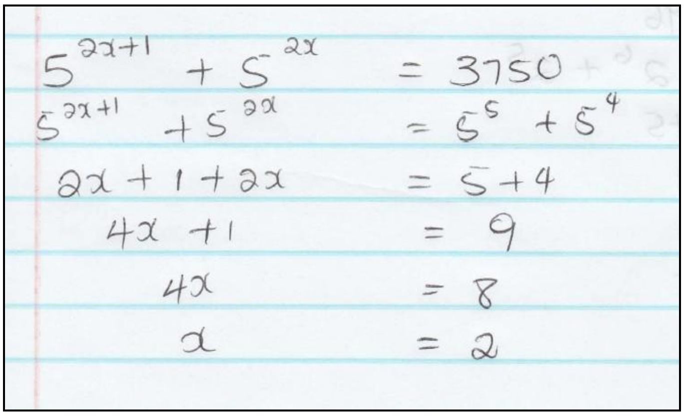

Figure 1. The Student's Unanticipated Approach

There is one thing in the student's approach that puzzled me, that is, concluding that:

If $5^{2x+1} + 5^{2x} = 5^5 + 5^4$, then $2x + 1 + 2x = 5 + 4$

There is no known theorem which supports this proposition. I was expecting the following approach:

$$5^{2x+1} + 5^{2x} = 3750$$
$$5^{2x}.5^{1} + 5^{2x} = 3750$$
$$5^{2x}(5 + 1) = 3750$$
$$5^{2x}(6) = 3750$$
$$5^{2x} = 625$$
$$5^{2x} = 5^{4}$$
$$\therefore\ 2x = 4$$
$$\Rightarrow\ x = 2$$

Figure 2. The Anticipated Approach

Clearly, it can be seen that the student's unanticipated approach gave the same answer as the anticipated approach. This prompted me to do further investigations. I presented both approaches to mathematics experts (through the Math Forum) for evaluation, and received the following comments:

COMMENT 1:

Doctor X: *The second approach is correct. The first is…um…not quite. In the second approach, each equation is logically equivalent to the one before it. Therefore, it is a valid mathematical argument (or proof) that the solution set contains exactly one solution, namely* $x = 2$.

In the first, the statement $5^{2x+1} + 5^{2x} = 5^5 + 5^4$ *is logically equivalent to the one before it. But there is no reasonable way to get from that statement to* $2x + 1 + 2x = 5 + 4$ *except perhaps by first proving that the only solution to the former is* $x = 2$. *If someone had instead gotten to* $5^{2x+1} + 5^{2x} = 5^5 + 5^4$ *and then said, 'And then if* $2x + 1 = 5$ *and* $2x = 4$, *then we have a solution!' then I would say, 'Yes! That is correct. And that proves that there are no other solutions'. But when you put the + in between those two pieces then there is no sense to this.*

COMMENT 2:

Doctor Y: *We're talking about this step*

:

$5^{2x+1} + 5^{2x} = 5^5 + 5^4$

$2x + 1 + 2x = 5 + 4$

In the absence of any explicit reason given for this step, I can only guess that whoever did this work was relying on a supposed theorem that:

$$x^a + x^b = x^c + x^d \Rightarrow a + b = c + d$$

A similar theorem is well known to be true, and a critical part of solving many exponential equations:

$$x^a = x^b \Leftrightarrow a = b$$

But then this cannot be extended to a SUM of exponentials with the same base, unless we can prove it. I doubt that it is true, but I'll need to look into it to be sure.

I could see that the experts' comments were far from being conclusive. The question of why the student's method which appeared to be invalid gave a correct answer in the end was not answered. Was it by coincidence or was there some hidden logic behind it? I decided to spent more time analyzing the student's approach to figure out why it gave a correct answer. After several hours of critical thinking, I discovered that there was a pattern in the manner in which the student wrote the exponents. I noticed that the difference between the exponents on the left-hand side was the same as the difference between the exponents on the right-hand side. I conjectured that this was the magic behind the student's correct answer. I then set out to conduct my own empirical investigations by applying the student's approach to several other similar exponential equations. Here are my findings:

Equation 1: Solve

$$5^{x+1} + 5^x = 6$$

Anticipated Approach	Student's Unanticipated Approach
$5^x.5 + 5^x = 6$	$5^{x+1} + 5^x = 5^1 + 5^0$
$5^x(5+1) = 6$	$x + 1 + x = 1 + 0$
$5^x(6) = 6$	$2x + 1 = 1$
$5^x = 1$	$2x = 0$
$5^x = 5^0$	$\therefore\ x = 0$
$\therefore\ x = 0$	$Note: (x+1) - x = 1\ and\ 1 - 0 = 1$

Equation 2: Solve

$$2^{x+1} + 2^x = 96$$

Anticipated Approach	Student's Unanticipated Approach
$2^x.2^1 + 2^x = 96$	$2^{x+1} + 2^x = 2^6 + 2^5$
$2^x(2+1) = 96$	$x + 1 + x = 6 + 5$
$2^x(3) = 96$	$2x + 1 = 11$
$2^x = 32$	$2x = 10$
$2^x = 2^5$	$\therefore\ x = 5$
$\therefore\ x = 5$	$Note: (x+1) - x = 1\ and\ 6 - 5 = 1$

Equation 3: Solve

$$2^{x} - 2^{x-2} = 3$$

Anticipated Approach	Student's Unanticipated Approach
$2^{x} - 2^{x}.2^{-2} = 3$ $2^{x}(1 - 2^{-2}) = 3$ $2^{x}\left(1 - \frac{1}{4}\right) = 3$ $2^{x}\left(\frac{3}{4}\right) = 3$ $2^{x} = 3 \times \frac{4}{3}$ $2^{x} = 4 = 2^{2}$ $\therefore\ x = 2$	$2^{x} - 2^{x-2} = 2^{2} - 2^{0}$ $x + x - 2 = 2 + 0$ $2x - 2 = 2$ $2x = 4$ $\therefore\ x = 2$ $Note: x - (x - 2) = 2 \text{ and } 2 - 0 = 2$

Equation 4: Solve

$$3^{x} + 3^{x+2} = 10$$

Anticipated Approach	Student's Unanticipated Approach
$3^{x}.3^{2} + 3^{x} = 10$ $3^{x}(3^{2} + 1) = 10$ $3^{x}(10) = 10$ $3^{x} = 1$ $3^{x} = 3^{0}$ $\therefore\ x = 0$	$3^{x+2} + 3^{x} = 3^{2} + 3^{0}$ $x + 2 + x = 2 + 0$ $2x + 2 = 2$ $2x = 0$ $\therefore\ x = 0$ $Note: (x + 2) - x = 2 \text{ and } 2 - 0 = 2$

Equation 5: Solve

$$3^{x+1} + 3^{x-1} = \frac{10}{9}$$

Anticipated Approach	Student's Unanticipated Approach
$3^{x}.3^{1} + 3^{x}.3^{-1} = \frac{10}{9}$ $3^{x}(3^{1} + 3^{-1}) = \frac{10}{9}$ $3^{x}\left(3 + \frac{1}{3}\right) = \frac{10}{9}$ $3^{x}\left(\frac{10}{3}\right) = \frac{10}{9}$ $3^{x} = \frac{10}{9} \times \frac{3}{10} = \frac{1}{3} = 3^{-1}$ $\therefore\ x = -1$	$3^{x+1} + 3^{x-1} = (3^{2} + 3^{0})3^{-2}$ $3^{x+1} + 3^{x-1} = 3^{0} + 3^{-2}$ $x + 1 + x - 1 = 0 + (-2)$ $2x = -2$ $\therefore\ x = -1$ $Note: (x + 1) - (x - 1) = 2 \text{ and } 0 - (-2) = 2$

VI. Discussionof Results

After trying out several cases, I concluded that it was not by coincidence that the student's unanticipated approach to solving exponential equations gave a correct answer. Indeed, the student's technique works under certain conditions. The only problem with the student's presentation was that it was not backed up by the known exponential laws and lacked the necessary details to assist the educator in marking thestudent's work. The student therefore risked being marked wrong because the logic behind the student's approach was not easily discernible. There is no known theorem which suggests that:

If $5^{2x+1} + 5^{2x} = 5^5 + 5^4$, then $2x + 1 + 2x = 5 + 4$.

Based on my findings from using the student's technique in several similar cases, I would like to propose the possibility of developing such a theorem, under certain restrictions:

Proposition: If $a^x \pm a^{x+p} = k$, where k can be written in the form $a^y \pm a^{y+p}$, then:

a) $x = y$ **(1)**

b) $x + p = y + p$ **(2)**

c) $x + x + p = y + y + p$ **(3)**

Note: $x - (x + p) = y - (y + p)$

Proof:

Suppose $a^x \pm a^{x+p} = k$, where $k = a^y \pm a^{y+p}$,

Then we have: $a^x \pm a^{x+p} = a^y \pm a^{y+p}$ (*)

$a^x \pm a^x . a^p = a^y \pm a^y . a^p$

$a^x(1 \pm a^p) = a^y(1 \pm a^p)$

$a^x = a^y$

$\therefore x = y$ **(1)**

If $a^x = a^y$,then from equation (*), $a^{x+p} = a^{y+p}$, which implies that:$x + p = y + p$ **(2)**

If results **(1)** and **(2)** are true, then it is also holds true that $x + x + p = y + y + p$ **(3)**

Result (3) explains why the student's unanticipated approach gave a correct answer in the end. I therefore argue here that my student's solution method is acceptable and I am inviting the mathematics community to criticize these findings.

VII. Recommendations

Based on the findings of this investigation, I strongly urge fellow mathematics educators to closely examine students' solutions to avoid marking them wrong when they are right. Some of the methods of solutions that students may use might be completely new and unfamiliar to the educator. Like the famous German mathematician Carl Friedrich Gauss (1777-1855) who discovered a quick way of adding natural numbers from 1 to 100 at the age of seven, these young minds should not be looked down upon. Some of the students we teach possess exceptional intellectual abilities that need to be nurtured and not suppressed. Good mathematics teachers should therefore be ready to learn even from their own students. Finally, I recommend further studies on the possibility of extending exponential laws to a sum or difference of exponentials.

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