Unit Graphs and Subgraphs of Finite Groups $U_n$ and $K_4$

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Abstract: We represent finite group in the form of graphs. These graphs are called unit graphs. In this paper we shall study unit graphs of finite groups $U_n$ (multiplicative group of integers modulo n) and $K_4$ (Klein’s four group). Also study of different properties like the subgroups of a group are carried out using the unit graph of the group.

Keywords: Finite Group, Graph, Subgraph

I. INTRODUCTION

The phenomenon of representing Groups using Graphs has been studied theoretically by number of researchers [6-8, 12, 13]. In a series of investigations S. Akbari and A. Mohammadian [1] discussed the zero divisor graphs of finite rings. In this paper, we give unit graphs and subgraphs of finite groups $U_n$ (multiplicative group of integers modulo n) and $K_4$ (Klein’s four group).

II. UNIT GRAPH OF A GROUP

2.1. Definition

A graph $G (V, E)$ is Unit Graph of a group $(G, .)$ if

(i) Distinct elements $v_i$ and $v_j$ are adjacent in graph $G (V, E)$ if $v_i . v_j = e$ in group $(G, .)$.

(ii) Every element of group $(G, .)$ is adjoined with the unity of group $(G,.).

2.2. Unit Graph of $U_n$

The multiplicative group of integers modulo n is given by $U_n = \{x \in \mathbb{Z} | 1 \leq x < n, (x, n) = 1\}$. The unit graphs of $U_n$ for some n are given as follows:

2.2.1. Unit Graph of $U_2$

The multiplicative group of integers modulo 2 is $U_2 = \{1\}$. The unit graph of $U_2$ is:

\begin{center}
\begin{tikzpicture}
\node (a) at (0,0) {$1$};
\end{tikzpicture}
\end{center}

\textbf{Fig. 1: Unit Graph of $U_2$}

We have some following properties of this graph:-

(i) This graph is finite graph.

(ii) This graph is complete graph.

(iii) This graph is connected graph

(iv) The chromatic number is $\chi(U_2) = 1$.

2.2.2. Unit Graph of $U_3$

The multiplicative group of integers modulo 3 is $U_3 = \{1,2\}$. The unit graph of $U_3$ is:

\begin{center}
\begin{tikzpicture}
\node (a) at (0,0) {$1$};
\node (b) at (0.3,0) {$2$};
\end{tikzpicture}
\end{center}

\textbf{Fig. 2: Unit Graph of $U_3$}

We have some following properties of this graph:-

(i) This graph is Bipartite because the vertex set V can be decomposed into two disjoint subsets $V_1$ and $V_2$ such that every edge in $U_3$ has one end point in $V_1$ and one end point in $V_2$.

(ii) This graph is Regular because every vertex is of same degree i.e., every vertex is of degree one.

(iii) This graph is finite because there are finite numbers of vertices and edges.

(iv) This graph is complete graph.

(v) This graph is connected graph

(vi) This graph is planar graph.

(vii) The chromatic number is $\chi(U_3) = 2$. 

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2.2.3. Unit Graph of $U_5$

The multiplicative group of integers modulo 5 is $U_5 = \{1,2,3,4\}$. The unit graph of $U_5$ is:

![Fig. 3: Unit Graph of $U_5$](image)

We have some following properties of this graph:

(i) This graph is finite graph.
(ii) This graph is connected graph.
(iii) The chromatic number is $\chi(U_5) = 3$.

2.2.4. Unit Graph of $U_9$

The multiplicative group of integers modulo 9 is $U_9 = \{1,2,4,5,7,8\}$. The unit graph of $U_9$ is:

![Fig. 4: Unit Graph of $U_9$](image)

We have some following properties of this graph:

(i) This graph is finite graph.
(ii) This graph is connected graph.
(iii) The chromatic number is $\chi(U_9) = 3$.

2.2.5. Unit Graph of $U_{13}$

The multiplicative group of integers modulo 13 is $U_{13} = \{1,2,3,4,5,6,7,8,9,10,11,12\}$. The unit graph of $U_{13}$ is:

![Fig. 5: Unit Graph of $U_{13}$](image)

We have some following properties of this graph:

(i) This graph is finite graph.
(ii) This graph is connected graph.
(iii) The chromatic number is $\chi(U_{13}) = 3$.

2.2.6. Unit Graph of $U_{16}$

The multiplicative group of integers modulo 16 is $U_{16} = \{1,3,5,7,9,11,13,15\}$. The unit graph of $U_{16}$ is:
We have some following properties of this graph:

(iv) This graph is finite graph.
(v) This graph is connected graph.
(vi) The chromatic number is $\chi(U_{16}) = 3$.

2.2.7. Unit Graph of $U_{17}$
The multiplicative group of integers modulo 17 is $U_{17} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$. The unit graph of $U_{17}$ is:

![Unit Graph of $U_{17}$]

We have some following properties of this graph:

(iv) This graph is finite graph.
(v) This graph is connected graph.
(vi) The chromatic number is $\chi(U_{17}) = 3$.

2.2.8. Unit Graph of $U_{19}$
The multiplicative group of integers modulo 19 is $U_{19} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$. The unit graph of $U_{19}$ is:

![Unit Graph of $U_{19}$]

We have some following properties of this graph:

(i) This graph is finite graph.
(ii) This graph is connected graph.
(iii) The chromatic number is $\chi(U_{19}) = 3$.

2.2.9. Unit Graph of $U_{21}$
The multiplicative group of integers modulo 21 is $U_{21} = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$. The unit graph of $U_{21}$ is:

![Unit Graph of $U_{21}$]
We have some following properties of this graph:-
(i) This graph is finite graph.
(ii) This graph is connected graph.
(iii) The chromatic number is $\chi(U_{21}) = 3$.

2.2.10. Unit Graph of $U_{22}$
The multiplicative group of integers modulo 22 is $U_{22} = \{1,3,5,7,9,13,15,17,19,21\}$. The unit graph of $U_{22}$ is:

We have some following properties of this graph:-
(i) This graph is finite graph.
(ii) This graph is connected graph.
(iii) The chromatic number is $\chi(U_{22}) = 3$.

2.3. Unit Graph of $K_4$
The Klein’s four group $K_4$ is given by $K_4 = \{e, a, b, c\}$, where $a^2 = b^2 = c^2 = e$ and $ab = ba = c, bc = cb = a$ and $ca = ac = b$. The unit graph of $K_4$ is:

We have some following properties of this graph:-
(i) This graph is finite graph.
(ii) This graph is connected graph.
(iii) This graph is planar graph.
(iv) The chromatic number is $\chi(K_4) = 2$.

III. UNIT SUBGRAPH OF A GROUP

3.1. Definition
Let graph $G (V, E)$ be an unit graph of a group $(G, \cdot)$. If $H$ is a subgroup of the group $G$, then the unit graph drawn for the subgroup $H$ is known as the unit subgraph of the group $G$.

3.2. Unit Subgraphs of $U_n$
The unit subgraphs of $U_n$ for some $n$ are given as follows:
3.2.1. Unit Subgraphs of $U_2$

The multiplicative group of integers modulo 2 is $U_2 = \{1\}$. Subgroup of $U_2$ is $H_1 = \{1\} = U_2$ only.

The unit subgraph of $U_2$ is:

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
\end{tikzpicture}
$$

Fig. 12: Unit Graph of $H_1$

3.2.2. Unit Subgraphs of $U_3$

The multiplicative group of integers modulo 3 is $U_3 = \{1, 2\}$. Subgroups of $U_3$ are $H_1 = \{1\}$ and $H_2 = \{1, 2\} = U_3$. The unit subgraphs of $U_3$ are:

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
\end{tikzpicture}
$$

Fig. 13: Unit Graph of $H_1$

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
\end{tikzpicture}
$$

Fig. 14: Unit Graph of $H_2$

3.2.3. Unit Subgraphs of $U_5$

The multiplicative group of integers modulo 5 is $U_5 = \{1, 2, 3, 4\}$. Subgroups of $U_5$ are $H_1 = \{1\}$, $H_2 = \{1, 4\}$ and $H_3 = \{1, 2, 3, 4\} = U_5$. The unit subgraphs of $U_5$ are:

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$3$};
\end{tikzpicture}
$$

Fig. 15: Unit Graph of $H_1$

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$3$};
\end{tikzpicture}
$$

Fig. 16: Unit Graph of $H_2$

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$3$};
\end{tikzpicture}
$$

Fig. 17: Unit Graph of $H_3$

3.2.4. Unit Subgraphs of $U_9$

The multiplicative group of integers modulo 9 is $U_9 = \{1, 2, 4, 5, 7, 8\}$. Subgroups of $U_9$ are $H_1 = \{1\}$, $H_2 = \{1, 8\}$, $H_3 = \{1, 4, 7\}$ and $H_4 = \{1, 2, 4, 5, 7, 8\} = U_9$. The unit subgraphs of $U_9$ are:

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$3$};
\end{tikzpicture}
$$

Fig. 18: Unit Graph of $H_1$

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$3$};
\end{tikzpicture}
$$

Fig. 19: Unit Graph of $H_2$

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$3$};
\end{tikzpicture}
$$

Fig. 20: Unit Graph of $H_3$

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
  \node (3) at (2,0) {$3$};
\end{tikzpicture}
$$

Fig. 21: Unit Graph of $H_4$

3.2.5. Unit Subgraphs of $U_{13}$

The multiplicative group of integers modulo 13 is $U_{13} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Subgroups of $U_{13}$ are $H_1 = \{1\}$, $H_2 = \{1, 12\}$, $H_3 = \{1, 3, 9\}$ and $H_4 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} = U_{13}$ etc. The unit subgraphs of $U_{13}$ are:

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
\end{tikzpicture}
$$

Fig. 22: Unit Graph of $H_1$

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
\end{tikzpicture}
$$

Fig. 23: Unit Graph of $H_2$

$$
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (1,0) {$2$};
\end{tikzpicture}
$$

Fig. 24: Unit Graph of $H_3$
3.2.6. Unit Subgraphs of $U_{16}$

The multiplicative group of integers modulo 16 is $U_{16} = \{1,3,5,7,9,11,13,15\}$. Subgroups of $U_{16}$ are $H_1 = \{1\}$, $H_2 = \{1,7\}$, $H_3 = \{1,9\}$, $H_4 = \{1,15\}$, $H_5 = \{1,5,9,13\}$ and $H_6 = \{1,3,5,7,9,11,13,15\} = U_{16}$ etc. The unit subgraphs of $U_{16}$ are:

![Unit Graph of $H_1$](image)

![Unit Graph of $H_2$](image)

![Unit Graph of $H_3$](image)

![Unit Graph of $H_4$](image)

![Unit Graph of $H_5$](image)

![Unit Graph of $H_6$](image)

3.2.7. Unit Subgraph of Unit Group $U_{17}$

The multiplicative group of integers modulo 17 is $U_{17} = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$. Subgroups of $U_{17}$ are $H_1 = \{1\}$, $H_2 = \{1,16\}$, $H_3 = \{1,2,4,8,9,13,15,16\}$ and $H_4 = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\} = U_{17}$ etc. The unit subgraphs of $U_{17}$ are:

![Unit Graph of $H_1$](image)

![Unit Graph of $H_2$](image)

![Unit Graph of $H_3$](image)

![Unit Graph of $H_4$](image)

3.2.8. Unit Subgraphs of Unit Group $U_{19}$

The multiplicative group of integers modulo 19 is $U_{19} = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18\}$. Subgroups of $U_{19}$ are $H_1 = \{1\}$, $H_2 = \{1,18\}$, $H_3 = \{1,7,11\}$ and $H_4 = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18\} = U_{19}$ etc. The unit subgraphs of $U_{19}$ are:
3.2.9. Unit Subgraphs of Unit Group $U_{21}$

The multiplicative group of integers modulo 21 is $U_{21} = \{1,2,4,5,8,10,11,13,16,17,19,20\}$. Subgroups of $U_{21}$ are $H_1 = \{1\}$, $H_2 = \{1,8\}$, $H_3 = \{1,13\}$, $H_4 = \{1,20\}$, $H_5 = \{1,8,13,20\}$ and $H_4 = \{1,2,4,5,8,10,11,13,16,17,19,20\} = U_{21}$ etc. The unit subgraphs of $U_{21}$ are:

3.2.10. Unit Subgraphs of Unit Group $U_{22}$

The multiplicative group of integers modulo 22 is $U_{22} = \{1,3,5,7,9,13,15,17,19,21\}$. Subgroups of $U_{22}$ are $H_1 = \{1\}$, $H_2 = \{1,21\}$, $H_3 = \{1,3,5,9,15\}$ and $H_4 = \{1,3,5,7,9,13,15,17,19,21\} = U_{22}$ etc. The unit subgraphs of $U_{22}$ are:
3.3. Unit Subgraphs of $K_4$

The Klein’s four group $K_4$ is given by $K_4 = \{e, a, b, c\}$, where $a^2 = b^2 = c^2 = e$ and $ab = ba = c$, $bc = cb = a$ and $ca = ac = b$. Subgroups of $K_4$ are $H_1 = \{e\}, H_2 = \{e, a\}, H_3 = \{e, b\}, H_4 = \{e, c\}$ and $H_5 = \{e, a, b, c\} = K_4$. The unit subgraphs of $K_4$ are:

Fig. 50: Unit Graph of $H_1$

Fig. 51: Unit Graph of $H_2$

Fig. 52: Unit Graph of $H_3$

Fig. 53: Unit Graph of $H_4$

Fig. 54: Unit Graph of $H_5$

REFERENCES