Integral Solutions of Non-Homogeneous Biquadratic Diophantine Equation \( x^4 - y^4 = 34 (z + w) p^2 \) With Five Unknowns  

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**Abstract:** The non-homogeneous biquadratic Diophantine equation given by \( x^4 - y^4 = 34 (z + w) p^2 \) is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.  

**Keywords:** Non-homogeneous Equation, Integral Solutions, Polygonal Numbers, Pyramidal Numbers and Special Numbers  

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**I. Introduction**  
Biquadratic Diophantine equations homogenous and non–homogenous are rich in variety [1-3]. For an extensive review of sizable literature and various problems, one may refer [4-18]. In this communication, we consider yet another interesting biquadratic equation \( x^4 - y^4 = 34 (z + w) p^2 \) and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions and special polygonal numbers are presented.

**Notations Used**  
- \( \tau_{m,n} \) - Polygonal number of rank ‘n’ with size ‘m’  
- \( CP_n^6 \) - Centered hexagonal Pyramidal number of rank n  
- \( Gn_n \) - Gnomic number of rank ‘n’  
- \( FN_4^A \) - Figurative number of rank ‘n’ with size ‘m’  
- \( Pr_n \) - Pronic number of rank ‘n’  
- \( P_m^n \) - Pyramidal number of rank ‘n’ with size ‘m’  
- \( carl_n \) -- Carol number  
- \( ky_n \) - Keynea number  
- \( Tha_n \) - Thabit number  
- \( S_n \) – Star number of rank n  
- \( SO_n \) - Stella octagonal number of rank n  

**II. Method of Analysis**  
The Diophantine equation representing the biquadratic equation with five unknowns to be solved for its non zero distinct integral solutions is  
\[ x^4 - y^4 = 34(z + w)p^2 \]  
Consider the transformation  
\[
\begin{align*}
x &= u + v \\
y &= u - v \\
z &= 2uv + 1 \\
p &= 2uv - 1
\end{align*}
\]  
On substituting (2) in (1), we get  
\[ u^2 + v^2 = 17 \ p^2 \]  
Now, We illustrate methods of obtaining non Zero distinct integer solutions to (1)  

**Pattern I**
Assume \[ p = A^2 + B^2 = (A + iB)(A - iB) \] (4)

Let \[ u^2 + v^2 = 16p^2 + p^2 \]

Employing the method of factorization, we get the system of equations

\[
\begin{align*}
  u(A, B) &= A^2 - B^2 + 8AB \\
  v(A, B) &= -4A^2 + 4B^2 + 2AB \\
  p(A, B) &= A^2 + B^2
\end{align*}
\]

Substituting the values of \( u, v \) and \( p \) in (2), the nonzero distinct integral values of \( x, y, z, w \) and \( p \) are given by

\[
\begin{align*}
x &= x(A, B) = -3A^2 + 3B^2 + 10AB \\
y &= y(A, B) = 5A^2 - 5B^2 + 6AB \\
z &= z(A, B) = -8A^4 - 8B^4 + 48A^2B^2 - 60A^3B + 60AB^3 + 1 \\
w &= w(A, B) = -8A^4 - 8B^4 + 48A^2B^2 - 60A^3B - 60AB^3 - 1 \\
p &= p(A, B) = A^2 + B^2
\end{align*}
\]

Properties

1. \( y(A, A) + x(A, A) - 2p(A, A) - 16t_{A, B} \equiv 0 \)
2. \( y(1,1) + x(1,1) - 2p(1,1) \) is a perfect square
3. \( 5x(A, 1) + 3y(A, 1) \equiv 0 \text{ (mod 68)} \)
4. \( z(A, B) - w(A, B) \) is a Thabit number
5. \( z(1,1) + w(1,1) \) is a cubic integer
6. \( -6[z(1,0) + w(1,0)] \) is a nasty number
7. \( 31[y(1,1) + p(1,1)] \) is a perfect number
8. \( z(A, 1) + w(A, 1) + 192F_{n_A} - 80P_{r_A} + 120C_{p_A} - 16Gno_A \equiv 0 \text{ (mod 8)} \)

Pattern II

Equation (3) as

\[ u^2 + v^2 = 17p^2 \]

Where \( 17 = (4 + i)(4 - i) \)

Using (4) and (5) in (3) and writing (3) in factorization form as

\[
(u + iv)(u - iv) = (4 + i)(4 - i)[(A + iB)(A - iB)]^2
\]

which is equivalent to the system of equation

\[ u + iv = (4A^2 - 4B^2 - 2AB) + i(A^2 - B^2 + 8AB) \]

Equating real and imaginary parts, we get

\[ u = 4A^2 - 4B^2 - 2AB \]
\[ v = A^2 - B^2 + 8AB \]

On substituting the values of \( u \) and \( v \) in (2) the nonzero distinct integral values of \( x, y, z, w \) and \( p \) satisfying (1) are given by

\[
\begin{align*}
x &= x(A, B) = 5A^2 - 5B^2 + 6AB \\
y &= y(A, B) = 3A^2 - 3B^2 - 10AB \\
z &= z(A, B) = 8A^4 + 8B^4 - 48A^2B^2 + 60A^3B - 60AB^3 + 1 \\
w &= w(A, B) = 8A^4 + 8B^4 - 48A^2B^2 + 60A^3B + 60AB^3 - 1 \\
p(A, B) &= A^2 + B^2
\end{align*}
\]

Properties

1. \( x(A, A) + y(A, A) + 8p(A, A) - 16P_{r_A} \equiv 0 \text{ (mod 20)} \)
2. \( x(1, B) + y(1, B) + 8p(1, B) \equiv 16 \text{ (mod 4)} \)
3. \( z(A, 1) + w(A, 1) - 192F_{n_A} - 60s_{o_A} + 60P_{r_A} + 20t_{A, A} - 16 \equiv 0 \)
4. \( z(1, B) + w(1, B) - 16B_{n_B} - 16S_{o_B} + 165B_{n_B} + 18Gno_{n_A} - 14 \equiv 0 \)
5. \( z(1, 1) + w(1, 1) \) is a cubic integer
6. \( z(1,0) + w(1,0) \) is a perfect number

Pattern III
Equation (3) as
\[ u^2 + v^2 = 17p^2 \]

Where \( 17 = (1 + 4i)(1 - 4i) \)

Using (4) and (6) in (3) and writing (3) in factorization form as
\[ (u + iv)(u - iv) = (1 + 4i)(1 - 4i)((A + iB)(A - iB))^2 \]

which is equivalent to the system of equation
\[ u + iv = (A^2 - B^2 - 8AB) + i(4A^2 - 4B^2 + 2AB) \]

Equating real and imaginary parts, we get
\[ u = A^2 - B^2 - 8AB \]
\[ v = 4A^2 - 4B^2 + 8AB \]

On substituting the values of \( u \) and \( v \) in (2) the nonzero distinct integral values of \( x, y, z, w \) satisfying (1) are given by
\[
\begin{align*}
x &= x(A, B) = 5A^2 - 5B^2 - 6AB \\
y &= y(A, B) = -3A^2 - 3B^3 - 10AB \\
z &= z(A, B) = 8A^4 + 8B^4 - 48A^2B^2 - 60A^3B + 60A^3B^3 + 1 \\
w &= w(A, B) = 8A^4 + 8B^4 - 48A^2B^2 - 60A^3B + 60A^3B^3 - 1 \\
p &= p(A, B) = A^2 + B^2
\end{align*}
\]

Properties
1. \( x(3,2) + y(3,2) + 2p(3,2) \) is a nasty number
2. \( z(A,1) + w(A,1) - 192FN_A^2 + 120CP_A + 300x_P - 30GnO_A - 46 \equiv 0 \)
3. \( y(1, -1) + 3p(1, -1) \) is a perfect square
4. \( 3x(A,1) + 5y(A,1) + 68A \equiv 0 \)
5. \( x(B, B) + p(B, B) + 2t_{4, \beta} \equiv 0 \)

Pattern IV
Equation (3) as
\[ u^2 + v^2 = 17p^2 + 1 \]

Write 1 as
\[ 1 = \frac{(A^2 - B^2 - 2iAB)(A^2 - B^2 + 2iAB)}{(A^2 + B^2)^2} \]

Using (4) and (8) in (3) and writing (3) in factorization form as
\[ (u + iv)(u - iv) = (1 + 4i)(1 - 4i)((A + iB)(A - iB))^2 \frac{(A^2 - B^2 - 2iAB)(A^2 - B^2 - 2iAB)}{(A^2 + B^2)^2} \]

which is equivalent to the system of equation
\[ u + iv = (1 + 4i)(A + iB)^2 \frac{(A^2 - B^2 - 2iAB)}{(A^2 + B^2)} \]

Equating real and imaginary parts, we get
\[ u = \frac{1}{(A^2 + B^2)}(A^4 + 4B^4 - 6A^2B^2 - 16A^3B + 16AB^3) \]
\[ v = \frac{1}{(A^2 + B^2)}(A^4 + 4B^4 - 24A^2B^2 + 4A^3B - 4AB^3) \]

On substituting the values of \( u \) and \( v \) in (2) the nonzero distinct integral values of \( x, y, z, w \) and \( p \) satisfying (1) are given by
\[
\begin{align*}
x &= x(A, B) = \frac{1}{A^2 + B^2}(5A^4 + 5B^4 - 30A^2B^2 - 12A^3B + 12A^3B^3) \\
y &= y(A, B) = \frac{1}{A^2 + B^2}(-3A^4 - 3B^4 + 18A^2B^2 - 20A^3B + 20AB^3) \\
z &= z(A, B) = \frac{2}{(A^2 + B^2)^2}(4A^8 + 4B^8 - 60A^7B - 112A^6B^2 + 420A^5B^3 + 280A^4B^4 + 420A^3B^5 - 112A^2B^6 + 60AB^7) + 1 \\
w &= w(A, B) = \frac{2}{(A^2 + B^2)^2}(4A^8 + 4B^8 - 60A^7B - 112A^6B^2 + 420A^5B^3 + 280A^4B^4 + 420A^3B^5 - 112A^2B^6 + 60AB^7) - 1 \\
p &= p(A, B) = A^2 + B^2
\end{align*}
\]

Properties
1. \( x(A, 0) - 5t_{4, A} \equiv 0 \)
2. \( z(1, 1) + w(1, 1) + x(1, 1) \) is a nasty number
3. \( z(0, B) - 8B_1 - 1 \equiv 0 \)
4. \( y(1, -1) + 3p(1, -1) \) is a perfect square
5. \( p(2^n, 2^n) = Cn + n + 2 \)
6. \( 2p(A, A) \) is a nasty number
Integral Solutions of Non-Homogeneous Biquadratic Diophantine Equation $x^4 - y^4 = 34(z + p)^2$

III. Conclusion

In this paper, we have presented four different patterns of non-zero distinct integer solutions of biquadratic Diophantine equation $x^4 - y^4 = 34(z + w)p^2$ and relations between solutions and special numbers are also obtained. To conclude, one may search for other patterns of solutions and their corresponding properties.

References

Journal Articles


Reference Books
