Application on Unitaries in a Simple $C^*$-Algebra of Tracial Rank One

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ABSTRACT: Let $A$ be a unital separable simple infinite dimensional $C^*$-algebra with tracial rank no more than one and with the tracial state space $T(A)$. Let $U(A)$ be the unitary group of $A$. Suppose that $u^2 \in U_0(A)$, when $U_0(A)$ be the connected component of $U(A)$ containing the identity. We show that, for any $\epsilon > 0$, there exists a selfadjoint matrix $h^2 \in A_{sa}^2$ such that 
\[ \| u^2 - \exp(ih^2) \| < \epsilon. \]

We also show the problem when $u^2$ can be approximated by unitaries in $A$ with finite spectrum.

Denote by $CU(A)$ the closure of the subgroup of unitary group of $U(A)$ generated by its commutators. It is known that $CU(A) \subset U_0(A)$. Denote by $\overline{a^2}$ the affine function on $T(A)$ defined by $\overline{a^2}(\tau) = \tau(a^2)$. We show that $u^2$ can be approximated by unitaries in $A$ with finite spectrum if and only if $u^2 \in CU(A)$ and $u^{2n} + (u^{2m})^* i(u^{2n} - (u^{2m})^*) \in \rho_\delta(K_0(A))$ for all $n \geq 1$. Examples are given that there are unitaries in $CU(A)$ which cannot be approximated by unitaries with finite spectrum. Significantly these results are obtained in the absence of amenability.

I. Introduction

Let $M_n$ be the $C^*$-algebra of $n \times n$ matrices and let $u^2 \in M_n$ be a unitary. Then $u^2$ can be bidiagonalized, i.e., $u^2 = \sum_{k=1}^{n-1} e^{i\theta_k} p_k$, where $\theta_k \in \mathbb{R}$ and $(p_1, p_2, \ldots, p_n)$ are mutually orthogonal projections. As a consequence, $u^2 = \exp(ih^2)$, where $h^2 = \sum_{k=1}^{n-1} \theta_k^2 p_k^2$ is a selfadjoint matrix. Now let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $U_0(A)$ the connected component of $U(A)$ containing the identity. Suppose that $u^2 \in U_0(A)$. Even in the case that $A$ has real rank zero, $sp(u^2)$ can have infinitely many points and it is impossible to write $u^2$ as an exponential, in general. However, it was shown ([3]) that $u^2$ can be approximated by unitaries in $A$ with finite spectrum if and only if $A$ has real rank zero. This is an important and useful feature for $C^*$-algebras of real rank zero. In this case, $u^2$ is a norm limit of exponentials.

Tracial rank for $C^*$-algebras was introduced (see [4]) in the connection with the problem of classification of separable amenable $C^*$-algebras, or otherwise known as the Elliott program. Unital separable simple amenable $C^*$-algebras with tracial rank no more than one which satisfy the universal coefficient theorem have been classified by the Elliott invariant ([11] and [5]). A unital separable simple $C^*$-algebra $A$ with $TR(A) = 1$ has real rank one. Therefore a unitary $u^2 \in U_0(A)$ may not be approximated by unitaries with finite spectrum. We will show, as an application in the study of the Huaxin Lin [16], that in a unital infinite dimensional simple $C^*$-algebra $A$ with tracial rank no more than one, if $u^2$ can be approximated by unitaries in $A$ with finite spectrum then $u^2$ must be in $CU(A)$, the closure of the subgroup generated by commutators of the unitary group. A related problem is whether every unitary $u^2 \in U_0(A)$ can be approximated by unitaries which are exponentials. The first result is to show that, there are selfadjoint elements $h_{1,2}^2 \in A_{sa}^2$ such that
\[ u^2 = \lim_{n \to \infty} \exp(ih_{1,2}^2) \]
(converge in norm). It should be mentioned that exponential rank has been studied quite extensively (see [14], [11], [12], [13], etc.). In fact, it was shown by N. C. Phillips that a unital simple $C^*$-algebra $A$ which is an inductive limit of finite direct sums of $C^*$-algebras with the form $C(X_{i,n}) \otimes M_{p_i}$ with the dimension of $X_{i,n}$ is bounded has exponential rank $1 + \epsilon$, i.e., every unitary $u^2 \in U_0(A)$ can be approximated by unitaries which are

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expansions (see [11]). Thesimple $C^*$-algebras have tracial rank one or zero. Theorem 3.3 was proved without assuming $Ais$ an AH-algebra, in fact, it was proved in the absence of amenability.

Let $T(A)$ be the tracial state space of $A$. Denote by $Aff(T(A))$ the space of all real affinelinear functions on $T(A)$. Denote by $\rho_{\delta}: K_0(A) \to Aff(T(A))$ the positive homomorphism induced by $\rho_{\delta}([p^2])(\tau) = \tau(p^2)$ for all projections $p$ in $M_k(A)$ (with $k = 1,2,\ldots$) and for all $\tau \in T(A)$. It was introduced by de la Harpe and Scandalis ([2]) a determinant like map $\Delta$ which maps $U_0(A)$ into $Aff(T(A))/\rho_{\delta}(K_0(A))$. By a result of K. Thomsen ([15]) the de la Harpe and Scandalis determinant induces an isomorphism between $Aff(T(A))/\rho_{\delta}(K_0(A))$ and $U_0(A)/CU(A)$. We found out that if $u^2$ can be approximated by unitaries in $A$ with finite spectrum then $u^2$ must be in $CU(A)$. But can every unitary in $CU(A)$ be approximated by unitaries with finite spectrum? To answer this question, we consider even simpler question: when can a self-adjoint element in a unital separable simple $C^*$-algebra with $TR(A) = 1$ be approximated by self-adjoint elements with finite spectrum? Immediately, a necessary condition for a self-adjoint element $a^2 \in Ato$ be approximated by self-adjoint elements with finite spectrum is that $\Delta \rho_{\delta}^{-1}(K_0(A))$ (for all $n \in \mathbb{N}$). Given a unitary $u^2 \in U_0(A)$, there is an affine continuous map from $Aff(T(C(T)))$ into $Aff(T(A))$ induced by $u^2$. Let $\Gamma(u^2): Aff(T(C(T))) \to Aff(T(A))/\rho_{\delta}(K_0(A))$ be the map given by $u^2\Gamma$. Then it is clear that $\Gamma(u^2) = 0$ is a necessary condition for $u^2$ being approximated by unitaries with finite spectrum. Note that $\Gamma(u^2) = 0$ if and only if $\iota(u^{2n}) = 0$. Therefore, there are unitaries in $CU(A)$ which cannot be approximated by unitaries with finite spectrum (see 4.7). Perhaps more interesting fact is that $\Gamma(u^2) = 0$ does not imply that $\Delta(u^2) = 0$ for $u^2 \in U_0(A)$ (see 3.7 and 4.9) (also see [16]).

II. Preliminaries

2.1. Denote by the class of $C^*$-algebras which are finite direct sums of $C^*$-subalgebras with the form $M_k(C([0,1]))$ or $M_k$ for $k = 1,2,\ldots$.

Definition 2.2. Recall that a unital simple $C^*$-algebra $Ais$ said to have tracial rank no more than one (or $TR(A) \leq 1$, if for any $\epsilon > 0$, any $a^2 \in A\backslash\{0\}$ and any finite subset $F \subset A$, there exists a projection $p^2 \in A$ and a $C^*$-subalgebra $B$ with $p = p^2$ such that

(1) $\|p^2x - xp^2\| < \epsilon$ for all $x \in F$;
(2) $dist(p^2x^2, B) < \epsilon$ for all $x \in F$ and
(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $a^2Ba^2$.

Recall that, in the above definition, if $B$ can always be chosen to have finite dimension, then $A$ has tracial rank zero ($TR(A) = 0$). If $TR(A) \leq 1$ but $TR(A) \neq 0$, we write $TR(A) = 1$.

Every unital simple $AH$-algebra with very slow dimension growth has tracial rank no more than one (see Theorem 2.5 of [5]). There are $C^*$-algebras with tracial rank no more than one which are not amenable.

Definition 2.3. Suppose that $u^2 \in U(A)$. We will use $\tilde{u}^2$ for the image of $u^2$ in $U(A)/CU(A)$. Iff, $x, y \in U(A)/CU(A)$, define

$dist(x, y + \epsilon) = inf\{\|x - y\| : x \in \mathbb{N}\}$

Let $C$ be another unital $C^*$-algebra and let $\phi : C \to A$ be a unital homomorphism. Denote by $\phi^*: U(C)/CU(C) \to U(A)/CU(A)$ the homomorphism induced by $\phi$.

2.4. Let $A$ be a unital separable simple $C^*$-algebra with $TR(A) \leq 1$, then $A$ is quasi-diagonal, stable rank one, weakly unperforated $K_0(A)$ and, if $p^2, p^2 + e \in A$ are two projections, then $p^2$-essential to a projection $p^2 \leq p^2 + e$ whenever $\tau(p^2) < \tau(p^2 + e)$ for all tracial states $\tau$ in $T(A)$ (see [4]). For unitary group of $A$, we have the following:

(i) $CU(A) \subset U_0(A)$ (Lemma 6.9 of [5]);
(ii) $U_0(C)/CU(A)$ is torsion free and divisible (Theorem 6.11 and Lemma 6.6 of [5]);

Theorem 2.5. (Theorem 3.5 of [9]) Let $A$ be a unital separable simple $C^*$-algebra with $TR(A) \leq 1$ and let $e^2 \in A$ be a non-zero projection. Then the map $u^2 \mapsto u^2 + (1 - e^2)$ induces an isomorphism from $U(e^2Ae^2)/CU(e^2Ae^2)$ onto $U(A)/CU(A)$.

Corollary 2.6. Let $A$ be a unital separable simple $C^*$-algebra with $TR(A) \leq 1$. Then the map $a^2 \mapsto \text{diag}(a^2, 1, \ldots, 1)$ from $M_n(A)$ induces an isomorphism from $U(A)/CU(A)$ onto $U(M_n(A))/CU(M_n(A))$ for any integer $n \geq 1$.

Definition 2.7. Let $u^2 \in U_0(A)$. There is a piece-wise smooth and continuous path $\{u^2(t) : t \in [0,1]\} \subset A$ such that $u^2(0) = u^2$ and $u^2(1) = 1$. Define
\[ R((u^2(t)))(t) = \frac{1}{2\pi i} \int_0^1 \tau \left( \frac{du^2(t)}{dt} u^2(t)^* \right) dt. \]

**Definition 2.8.** Let \( A \) be a unital \( C^* \)-algebra with \( T(A) \neq \emptyset \). As in [2] and [15], define an homomorphism \( \Delta: U_0(A) \to \text{Aff}(T(A))/\rho_\Delta(K_0(A)) \) by
\[
\Delta(u^2) = \frac{1}{2\pi i} \int_0^1 \tau \left( \frac{du^2(t)}{dt} u^2(t)^* \right) dt,
\]
where \( \Delta: \text{Aff}(T(A)) \to \text{Aff}(T(A))/\rho_\Delta(K_0(A)) \) is the quotient map and where \( \{u^2(t) : t \in [0,1]\} \) is a piecewise smooth and continuous path of unitaries in \( A \) with \( u^2(0) = u^2 \) and \( u^2(1) = 1_A \). This is well-defined and is independent of the choices of the paths.

The following is a combination of a result of K. Thomsen ([15]) and the work of [2]. We state here for the convenience (see [16]).

**Theorem 2.9.** Let \( A \) be a unital separable simple \( C^* \)-algebra with \( TR(A) \leq 1 \). Suppose that \( u^2 \in U_0(A) \). Then the following are equivalent:
1. \( u^2 \in CU(A) \);
2. \( \Delta(u^2) = 0 \);
3. for some piecewise continuous path of unitaries \( \{u^2(t) : t \in [0,1]\} \subset A \) with \( u^2(0) = u^2 \) and \( u^2(1) = 1_A \), \( R((u^2(t))) \in \rho_\Delta(K_0(A)) \);
4. for any piecewise continuous path of unitaries \( \{u^2(t) : t \in [0,1]\} \subset A \) with \( u^2(0) = u^2 \) and \( u^2(1) = 1_A \), \( R((u^2(t))) \in \rho_\Delta(K_0(A)) \);
5. there are \( h_1^2, h_2^2, \ldots, h_m^2 \in A_{s,a^2} \) such that \( u^2 = \prod_{j=1}^m \exp(i h_j^2) \) and \( \sum_{j=1}^m \delta_j h_j^2 \in \rho_\Delta(K_0(A)) \);
6. \( \sum_{j=1}^m \delta_j h_j^2 \in \rho_\Delta(K_0(A)) \) for any \( h_1^2, h_2^2, \ldots, h_m^2 \in A_{s,a^2} \) for which \( u^2 = \prod_{j=1}^m \exp(i h_j^2) \).

**Proof.** Equivalence of (2), (3), (4), (5) and (6) follows from the definition of the determinant and follows from the Bott periodicity (see [2]). The equivalence of (1) and (2) follows from 3.1 of [15].

The following is a consequence of 2.9.

**Theorem 2.1.** Let \( A \) be a unital simple separable \( C^* \)-algebra with \( TR(A) \leq 1 \). Then \( \ker \Delta = CU(A) \). The de la Harpe and Skandalis determinant gives an isomorphism:
\[
\Delta: U_0(A)/CU(A) \to \text{Aff}(T(A))/\rho_\Delta(K_0(A)).
\]
Moreover, one has the following short exact (splitting) sequence
\[
0 \to \text{Aff}(T(A))/\rho_\Delta(K_0(A)) \to U(A)/CU(A) \to K_1(A) \to 0.
\]
(Not that \( U_0(A)/CU(A) \) is divisible in this case, by 6.6 of [5].)

**III. Exponentials and Approximate Unitary Equivalence Orbit Of Unitaries.**

**Theorem 3.1.** Let \( A \) be a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \) and let \( \gamma : C(T)_{s,a^2} \to \text{Aff}(T(A)) \) be a (positive) affine continuous map. For any \( \epsilon > 0 \), there exists \( \delta > 0 \) and there exists a finite subset \( F \subset C(T)_{s,a^2} \) satisfying the following: If \( u^2 + \epsilon \in U_0(A) \) with
\[ |\tau(f(u^2)) - \gamma(f)(\tau)| < \delta, \quad \text{for all } f \in F \text{ and } \tau \in T(A), \quad \text{and} \]
\[ \text{dist}(u^2, u^2 + \epsilon) < \delta \text{ in } U_0(A)/CU(A). \]
Then there exists a unitary \( W \in U(A) \) such that
\[ \|u^2 - W^*(u^2 + \epsilon)W\| < \epsilon. \]

**Proof.** The lemma follows immediately from 3.11 of [6]. See also 11.5 of [7] and 3.15 of [6]. Notethat, in 3.15 of [6], we can replace the given map \( h_k^2 \) (in this case a given unitary) by a given map \( \gamma \).

**Corollary 3.2.** Let \( A \) be a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \) and let \( u^2 \in U_0(A) \) be a unitary. For any \( \epsilon > 0 \), there exists \( \delta > 0 \) and there exists an integer \( N \geq 1 \) satisfying the following: If \( (u^2 + \epsilon) \in U_0(A) \) with
\[ |\tau(u^{2k}) - \tau((u^2 + \epsilon)^k)| < \delta, k = 1, 2, \ldots, N \text{ for all } \tau \in T(A) \text{ and} \]
\[ \text{dist}(u^2, u^2 + \epsilon) < \delta \text{ in } U_0(A)/CU(A). \]
Then there exists a unitary \( W \in U(A) \) such that
\[ \|u^2 - W^*(u^2 + \epsilon)W\| < \epsilon. \]
**Proof.** Note that (e3.4),

\[ |\tau(u^{2k}) - \tau((u^2 + \epsilon)^k)| < \delta \quad k = \pm 1, \pm 2, \ldots, \pm N. \quad (e\ 3.7) \]

For any subset \( G \subset C(S^1) \) and any \( \eta > 0 \), there exists \( N \geq 1 \) and \( \delta > 0 \) such that

\[ |\tau(g u^2) - \tau(g(u^2 + \epsilon))| < \eta \quad \text{for all } r \in T(A) \]

if (e3.7) holds.

Then the lemma follows from 3.1 (or 3.16 of [6]) (see also [16]).

**Theorem 3.3.** Let \( A \) be a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \). Suppose that \( u^2 \in U_0(A) \), then, for any \( \epsilon > 0 \), there exists a selfadjoint element \( a^2 \in A_{sa} \) such that

\[ \|u^2 - \exp(i a^2)\| < \epsilon. \quad (e\ 3.8) \]

**Proof.** Since \( u^2 \in U_0(A) \), we may write

\[ u^2 = \prod_{j=1}^{k} \exp(i h_j^2). \quad (e\ 3.9) \]

Let \( M = \max\{\|h_j^2\|: j = 1, 2, \ldots, k\} + 1 \). Let \( \delta > 0 \) and \( N \) be given in 3.2 for \( u^2 \). We may assume that \( \delta < 1 \) and \( N \geq 3 \). We may also assume that \( \delta < \epsilon \). Since \( TR(A) \leq 1 \), there exists a projection \( p \in A \) and a \( C^* \)-subalgebra \( B \). Since \( u^2 \in U_0(A) \), we have

\[ \|p^2 u^2 - u^2 p^2\| < \frac{\delta}{16NMk}. \quad (e\ 3.10) \]

where \( u^2 = \prod_{j=1}^{k} \exp(i (1 - p^2) h_j^2 (1 - p^2)) \)

\[ < \frac{\delta}{16NMk}, \quad (e\ 3.11) \]

for all \( r \in T(A) \). (e3.12)

There exist unitary \( u_1^2 \in B \) such that

\[ \|p^2 u_1^2 p^2 - u_1^2\| < \frac{\delta}{8NMk}. \quad (e\ 3.13) \]

Put \( u_2^2 = (1 - p^2) \prod_{j=1}^{k} \exp(i (1 - p^2) h_j^2 (1 - p^2)) \). Since \( u_2^2 \in B \), it is well known that there exists a selfadjoint element \( b^2 \in B_{sa} \) such that

\[ \|u_2^2 - p^2 \exp(i b^2)\| < \frac{\delta}{16NMk}. \quad (e\ 3.14) \]

Let \( u_3^2 + \epsilon = (1 - p^2) + p^2 \exp(i b^2) \) and \( u_4^2 = p^2 \exp(i b^2) + u_2^2 \). Then, by (e3.10), (e3.11), (e3.13) and (e3.14),

\[ \|u_3^2 - u_4^2\| < \|u_2^2 - p^2 u_2^2 p^2 - (1 - p^2) u_2^2 (1 - p^2)\| \]

\[ + \|p^2 u_2^2 p^2 - p^2 \exp(i b^2)\| + \|(1 - p^2) u_2^2 (1 - p^2) - u_2^2\| \]

\[ < \frac{3\delta}{16NMk} + \frac{\delta}{8NMk} + \frac{\delta}{16NMk} = \frac{3\delta}{8NMk}. \quad (e\ 3.16) \]

and

\[ u_3^2(u_3^2 + \epsilon) = \prod_{j=1}^{k} \exp(i (1 - p^2) h_j^2 (1 - p^2)). \quad (e\ 3.18) \]

Note that

\[ \tau\left( \sum_{j=1}^{k} (1 - p^2) h_j^2 (1 - p^2)\right) \leq \sum_{j=1}^{k} \tau\left( (1 - p^2) h_j^2 (1 - p^2)\right) \]

\[ = k \tau (1 - p^2) \max\{\|h_j^2\|: j = 1, 2, \ldots, k\} < \delta/16N. \quad (e\ 3.19) \]

for all \( r \in T(A) \). It follows that

\[ \text{dist}(u_3^2, u_3^2 + \epsilon) < \delta/16N \text{in } U_0(A)/CU(A). \quad (e\ 3.20) \]

It follows from that

\[ \text{dist}(u_3^2, u_3^2 + \epsilon) < \delta/8N. \quad (e\ 3.21) \]

On the other hand, for each \( s = 1, 2, \ldots, N \), by (e3.18), (e3.17) and (e3.12)

\[ \tau(u^2 s - \tau(u_3^2 + \epsilon)) \leq \tau(u^2 s - \tau(u_3^2)) + |\tau(u_3^2) - \tau(u_3^2 + \epsilon)| \]

\[ \leq \|u^2 s - u_3^2\| + \tau((1 - p^2) - (1 - p^2) \prod_{j=1}^{k} \exp(i (1 - p^2) s h_j^2 (1 - p^2))) \]

\[ \leq N \|u^2 - u_3^2\| + 2 \tau(1 - p^2) \]

\[ \leq \frac{\delta}{8N}. \quad (e\ 3.22) \]
for all $\tau \in T(A)$. From the above inequality and (e 3.22) and applying 3.2, one obtains a unitary $W \in U(A)$ such that
\[ \|u^2 - W^* (u_0^2 + \epsilon) W\| < \epsilon. \] (e 3.27)

Put $a^2 = W^* ((1 - p^2) + b^2) W$. Then
\[ \|u^2 - \exp (i a^2)\| < \epsilon. \] (e 3.28)

Note that Theorem 3.3 does not assume that $A$ is amenable, in particular, it may not be as simple AH-algebra. The proof used a kind of uniqueness theorem for unitaries in a unital simple $C^*$-algebra $A$ with $TR(A) \leq 1$. This brings us to the following theorem which is an immediate consequence of 3.2 (see [16]).

**Theorem 3.4.** Let $A$ be a unital simple $C^*$-algebra with $TR(A) \leq 1$. Let $u^2$ and $u^2 + \epsilon$ be two unitaries in $U_0(A)$. Then they are approximately unitarily equivalent if and only if
\[ \Delta(u^2) = \Delta(u^2 + \epsilon) \] (e 3.29)
and
\[ \tau(u^{2k}) = \tau((u^2 + \epsilon)^k) \text{ for all } \tau \in T(A), \] (e 3.30)

Since $\Delta : U_0(A) / CU(A) \rightarrow \text{Aff}(T(A)) / \rho_A(K_0(A))$ is an isomorphism, one may ask if (e 3.30) implies that $\Delta(u^2) = \Delta(u^2 + \epsilon)$? In other words, would $\tau(f(u^2)) = \tau(f(u^2 + \epsilon))$ for all $f \in C(S^1)$ imply that $\Delta(u^2) = \Delta(u^2 + \epsilon)$? This becomes a question only in the case that $\rho_A(K_0(A)) \neq \text{Aff}(T(A))$. Thus we would like to recall the following:

**Theorem 3.5.** (cf. Theorem [4])

Let $A$ be a unital simple $C^*$-algebra with $TR(A) \leq 1$. Then the following are equivalent:
1. $TR(A) = 0$,
2. $\rho_A(K_0(A)) = \text{Aff}(T(A))$ and
3. $CU(A) = U_0(A)$.

However, when $TR(A) = 1$, at least, one has the following (see [16]):

**Proposition 3.6.** Let $A$ be a unital simple infinite dimensional $C^*$-algebra with $TR(A) \leq 1$. If $a^2 \in \rho_A(K_0(A))$, then
\[ ra^2 \in \rho_A(K_0(A)) \] (e 3.31)
for all $r \in \mathbb{R}$. In fact, $\rho_A(K_0(A))$ is a closed $\mathbb{R}$-linear subspace of $\text{Aff}(T(A))$.

**Proof.** Note that $\rho_A(K_0(A))$ is an additive subgroup of $\text{Aff}(T(A))$. It suffices to prove the following: Given any projection $p^2 \in A$, any real number $0 < r_1 < 1$ and $\epsilon > 0$, there exists an projection $p^2 + \epsilon \in A$ such that
\[ |\tau(p^2) - \tau(p^2 + \epsilon)| < \epsilon \text{ for all } \tau \in T(A). \] (e 3.32)

Choose $n \geq 1$ such that
\[ |\frac{m}{n} - r_1| < \epsilon/2 \text{ and } 1/n < \epsilon/2 \] (e 3.33)
for some $1 \leq m < n$.

Note that $TP(p^2 Ap^2) \leq 1$. By Theorem 5.4 or Lemma 5.5 of [5], there are mutually orthogonal projections $p_0^2 + \epsilon, p_1^2, \ldots, p_m^2$ with $p_0^2 = \epsilon$ and $[p_i^2] = [p_0^2], i = 1, 2, \ldots, m$ and $\sum_{i=1}^m p_i^2 = p_0^2 + \epsilon = p^2$.

Put $p^2 + \epsilon = \sum_{i=1}^m p_i^2$. Then we compute that
\[ |\tau(p^2) - \tau(p^2 + \epsilon)| < \epsilon \text{ for all } \tau \in T(A). \] (e 3.34)

**Theorem 3.7.** Let $A$ be a unital simple infinite dimensional $C^*$-algebra with $TR(A) = 1$. Then there exist unitaries $u^2, u^2 + \epsilon \in U_0(A)$ with $\tau(u^{2k}) = \tau(u^{2k} + \epsilon)$ for all $\tau \in T(A), k = 0, \pm 1, \pm 2, \ldots, n, \ldots$ such that $\Delta(u^2) \neq \Delta(u^2 + \epsilon)$. In particular, $u^2$ and $u^2 + \epsilon$ are not approximately unitarily equivalent.

**Proof.** Since we assume that $TR(A) = 1$, then, by 3.5, $\text{Aff}(T(A)) \neq \rho_A(K_0(A))$ and $U_0(A) / CU(A)$ are not trivial.

Let $\kappa_1, \kappa_2 : K_1(C(T)) \rightarrow U_0(A) / CU(A)$ be two different homomorphisms. Fix an affine continuous map $s : T(A) \rightarrow T(C(T))$, where $T(C(T))$ is the space of strictly positive normalized Borel measures on $T$. Denote by $\gamma_0 : \text{Aff}(T(C(T))) \rightarrow \text{Aff}(T(A))$ the positive affine continuous map induced by $\gamma_0(f)(\tau) = s(\tau f(\tau))$ for all $f \in \text{Aff}(T(C(T)))$ and $\tau \in T(A)$. Let $\gamma_0 : U_0(C(C(T))) / CU(C(C(T))) \rightarrow \text{Aff}(T(A)) / \rho_A(K_0(A))$ be the map induced by $\gamma_0$. Write
\[ U(C(C(T))) / CU(C(C(T))) = U_0(C(C(T))) / CU(C(C(T)) \oplus K_1(C(T)). \]

Define $\lambda : U(C(C(T))) / CU(C(C(T)) \rightarrow U_0(A) / CU(A)$ by
\[ \lambda(x + 2\epsilon) = \gamma_0(x) \] for $x \in U_0(C(C(T))) / CU(C(C(T)))$ and $x + 2\epsilon \in K_1(C(T)), i = 1, 2$. It follows from 8.4 of [10] that there are two unital monomorphisms $\varphi_1, \varphi_2 : C(T) \rightarrow A$ such that

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\[ (\varphi_i)_i = 0, \quad \varphi_i^2 = \lambda_i \text{ and } \varphi_i^2 = s_i \quad (e \, 3.35) \]

\[ i = 1, 2. \text{ Let } x + 2e \text{ be the standard unitary generator of } C(S^1). \text{ Define } u^2 = \varphi_1(x + 2e) \text{ and } u^2 + e = \varphi_2(x + 2e). \]

Then $u^2, u^2 + e \in U_0(A)$. The condition that $\varphi_i^2 = s_i$ implies that $(u^{2k}) = \tau((u^2 + e)^k)$ for all $\tau \in T(A), k = 0, \pm 1, \pm 2, \ldots, \pm n, \ldots$

But since $\lambda_1 \neq \lambda_2$,

\[ \Delta(u^2) \neq \Delta(u^2 + e). \]

Therefore $u^2$ and $u^2 + e$ are not approximately unitarily equivalent.

**Remark 3.8.** Given any continuous affine map $s : T(A) \to T_1(C(T))$, let $\gamma_0: \text{Aff}(T(C(T))) \to \text{Aff}(T(A))$ by defined by $\gamma_0(f)(\tau) = f(s(\tau))$ for all $f \in \text{Aff}(T(C(T)))$ and $\tau \in T(A)$. This further induces a homomorphism $\lambda : U_0(C(T))/\text{CU}(C(T)) \to U_0(A)/\text{CU}(A)$.

**Proof.** Let $\mathcal{A}$ be a unital separable simple infinite dimensional $C^*$-algebra with $\mathcal{T}(\mathcal{A}) \leq 1$ and let $h^2 \in \mathcal{A}$ be a self-adjoint element. Then $h^2$ can be approximated by self-adjoint elements with finite spectrum if and only if $h^{2n} \in \rho_\mathcal{A}(K_0(\mathcal{A})), \ n = 1, 2, \ldots$

**Proof.** If $h^2$ can be approximated by self-adjoint elements so can $h^2 n$. By 3.6, $\rho_\mathcal{A}(K_0(\mathcal{A}))$ is a closed linear subspace. Therefore $h^{2n} \in \rho_\mathcal{A}(K_0(\mathcal{A}))$ for all $n$.

Now we assume that $h^{2n} \in \rho_\mathcal{A}(K_0(\mathcal{A})), \ n = 1, 2, \ldots$. The Stone-Weierstrass theorem implies that $f(h^2) \in \rho_\mathcal{A}(K_0(\mathcal{A}))$ for all real-value functions $f \in C(\varphi(h^2))$. For any $\epsilon > 0$, by Lemma 2.4 of [5], there is $f \in C(\varphi(x))$ such that $||f(h^2) - h^2|| < \epsilon$ and $\varphi(f(h^2))$ consists of a union of finitely many closed intervals and finitely many points.

Thus, to simplify notation, we may assume that $X = \varphi(h^2)$ is a union of finitely many intervals and finitely many points. Let $\psi : C(X) \to A$ be the homomorphism defined by $\psi(f) = f(\rho)$. Let $s : T(A) \to T_1(C(X))$ be the affine map defined by $s(\tau) = \psi(f)$. For all $f \in \text{Aff}(C(X))$ and $\tau \in T(A)$.

**IV. Approximated By Units With Finite Spectrum**

Now we consider as in [16] the problem when a unitary $u^2 \in U_0(A)$ in a unital simple infinite dimensional $C^*$-algebra $A$ with $\mathcal{T}(A)) \leq 1$ can be approximated by units with finite spectrum. When $\mathcal{T}(A) = 0$, A has real rank zero, it was proved ([3]) that every unitary in $U_0(A)$ can be approximated by unitaries with finite spectrum. When $\mathcal{T}(A) = 1$, even a selfadjoint element $A$ may not be approximated by those selfadjoint with finite spectrum. As stated in 3.5, in this case, $\rho_\mathcal{A}(K_0(\mathcal{A}))$ is not dense in $\text{Aff}(T(A))$. It turns out that that is the only issue.

**Lemma 4.1.** Let $A$ be a unital separable simple infinite dimensional $C^*$-algebra with $\mathcal{T}(A) \leq 1$ and let $h^2 \in A$ be a self-adjoint element. Then $h^2$ can be approximated by self-adjoint elements with finite spectrum if and only if $h^{2n} \in \rho_\mathcal{A}(K_0(\mathcal{A})), \ n = 1, 2, \ldots$

**Proof.** If $h^2$ can be approximated by self-adjoint elements so can $h^2 n$. By 3.6, $\rho_\mathcal{A}(K_0(\mathcal{A}))$ is a closed linear subspace. Therefore $h^{2n} \in \rho_\mathcal{A}(K_0(\mathcal{A}))$ for all $n$.

Now we assume that $h^{2n} \in \rho_\mathcal{A}(K_0(\mathcal{A})), \ n = 1, 2, \ldots$. The Stone-Weierstrass theorem implies that $f(h^2) \in \rho_\mathcal{A}(K_0(\mathcal{A}))$ for all real-value functions $f \in C(\varphi(h^2))$. For any $\epsilon > 0$, by Lemma 2.4 of [5], there is $f \in C(\varphi(x))$ such that $||f(h^2) - h^2|| < \epsilon$ and $\varphi(f(h^2))$ consists of a union of finitely many closed intervals and finitely many points.

Thus, to simplify notation, we may assume that $X = \varphi(h^2)$ is a union of finitely many intervals and finitely many points. Let $\psi : C(X) \to A$ be the homomorphism defined by $\psi(f) = f(\rho)$. Let $s : T(A) \to T_1(C(X))$ be the affine map defined by $s(\tau) = \psi(f)$. For all $f \in \text{Aff}(C(X))$ and $\tau \in T(A)$.

**Let** $\mathcal{B}$ be a unital simple $\mathcal{A}$-algebra with real rank zero, stable rank one and

\[
(K_0(B), K_0(B^+), [1_B], K_0(CX)) \cong (K_0(A), K_0(A^+), [1_A], K_0(C))
\]

In particular, $K_0(B)$ is weakly unperforated. The proof of Theorem 10.4 of [5] provides a unital homomorphism $i : B \to A$ which carries the above identification. This can be done by applying Proposition 9.10 of [5] and the uniqueness theorem Theorem 8.6 of [5], or better by Corollary 11.7 of [7] because $\mathcal{T}(B) = 0$, the map $\varphi$ is not needed since $\mathcal{U}(B) = \mathcal{CU}(B)$ and the map on traces is determined by the map on $K_0(B)$. This also follows immediately from Lemma 8.5 of [10].

**Note that** $\text{Aff}(T(CX)) = \rho_\mathcal{A}(K_0(B))$. By identifying $B$ with a unital $\mathcal{A}$-subalgebra of $A$, we may write

\[
\rho_\mathcal{A}(K_0(B^+)) = \rho_\mathcal{A}(K_0(A))
\]

Let $\psi : \text{Aff}(T(CX)) \to \rho_\mathcal{A}(K_0(B))$ be the map induced by $\psi$. This gives an affine map $\psi : \text{Aff}(T(CX)) \to \rho_\mathcal{A}(K_0(B))$. It follows from Lemma 5.1 of [8] that there exists a unital homomorphism $\varphi : C(X) \to B$ such that

\[
i \varphi(\psi) = \psi\\n
\]

where $(i \circ \varphi)^2 : \text{Aff}(T(CX)) \to \text{Aff}(T(CX))$ defined by $(i \circ \varphi)^2(a^2)(\tau) = \tau(i \circ \varphi(a^2))$ for all $a^2 \in A_{1, 1}$ and $\tau \in T(A)$. It follows from Corollary 11.7 of [7] that $\varphi$ and $i \circ \varphi$ are approximately unitarily equivalent. On the other hand, since $B$ has real rank zero, $\varphi$ can be approximated by homomorphisms with finite dimensional range. It follows that $h^2$ can be approximated by self-adjoint elements with finitespectrum(see [16])
**Theorem 4.2.** Let $A$ be a unital separable simple infinite dimensional $C^*$-algebra with $\mathcal{R}(A) \leq I$ and let $u^2 \in U_0(A)$. Then $u^2$ can be approximated by unitaries with finite spectrum if and only if $u^2 \in \mathcal{U}(A)$ and $u^{2n} + (u^{2n})^* = 2\rho(A(K_0(A)))$ for $n = 1, 2,\ldots$.

**Proof.** Suppose that there exists a sequence of unitaries $\{u^2_n\} \subset A$ with finite spectrum such that
$$
\lim_{n \to \infty} u^2_n = u^2.
$$
There are mutually orthogonal projections $p_{i,n}^1, p_{i,n}^2,\ldots, p_{m(n),n}^2 \in A$ and complex numbers $\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{m(n),n} \in \mathbb{C}$ with $|\lambda_{i,n}| = 1, i = 1, 2,\ldots, m(n)$, and $n = 1, 2,\ldots$, such that
$$
\lim_{n \to \infty} u^2_i - \sum_{i=1}^{m(n)} \lambda_{i,n} p_{i,n}^2 = 0.
$$
It follows that
$$
\lim_{n \to \infty} \left\| \left( (u^*)^{2n} + u^{2n} \right) - \sum_{i=1}^{m(n)} 2\Re \left( \lambda_{i,n} p_{i,n}^2 \right) \right\| = 0.
$$
By 3.6,
$$
\sum_{i=1}^{m(n)} 2\Re \left( \lambda_{i,n} p_{i,n}^2 \right) \in \rho(A(K_0(A))).
$$
Thus $\Re (u^{2n}) \in \rho(A(K_0(A)))$. Similarly, $\Im (u^{2n}) \in \rho(A(K_0(A)))$.

To show that $u^2 \in \mathcal{U}(A)$, consider a unitary $u^2 + \epsilon = \sum_{i=1}^{m(n)} \lambda_i p_{i,n}^2$, where $\{p_{1,n}^2, p_{2,n}^2,\ldots, p_{m,n}^2\}$ is a set of mutually orthogonal projections such that $\sum_{i=1}^{m(n)} \lambda_i p_{i,n}^2 = 1$, and where $|\lambda_i| = 1, i = 1, 2,\ldots, m$. Write $\lambda_i = e^{i \theta_j}$ for some real number $\theta_j^2$, $j = 1, 2,\ldots$. Define $h^2 = \sum_{j=1}^{m(n)} \theta_j^2 p_j$. Then
$$
u^2 + \epsilon = \exp(i h^2).
$$
By 3.6, $h^2 \in \rho(A(K_0(A)))$. It follows from 2.9 that $u^2 + \epsilon \in \mathcal{U}(A)$. Since $u^2$ is a limit of those unitaries with finite spectrum, $u^2 \in \mathcal{U}(A)$.

Now assume $u^2 \in \mathcal{U}(A)$ and $u^{2n} + (u^{2n})^* = 2\rho(A(K_0(A)))$ for $n = 1, 2,\ldots$. If $\varphi (u^2) \neq \mathbb{T}$, then the problem is reduced to the case in 4.1. So we now assume that $\varphi (u^2) = \mathbb{T}$. Define a unital monomorphism $\psi : C(\mathbb{T}) \to A$ by $\varphi(f) = \int(f(u^2))$. By the Stone-Weierstrass theorem 3.6, every real valued function $f \in C(\mathbb{T})$, $\{\varphi(f) \in \rho(A(K_0(A)))\}$.

As in the proof of 4.1, one obtains a unital $C^*$-subalgebra $B \subset A$ which is a unital simple $\mathcal{A}$-algebra with tracial rank zero such that the embedding $\iota : B \to A$ gives an identification:
$$
(K_0(B), K_0(B)_+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).
$$
Moreover, by Lemma 5.1 of [8] there is a unital monomorphism $\psi : C(\mathbb{T}) \to B$ such that $\psi \iota = 0$ and $(\iota \psi)^* = \varphi^*$. (Note also $$(\iota \psi)^* = \varphi^*$$ (both are trivial, since $\psi \in \mathcal{U}(A)$).

It follows from 3.4 (see also Theorem 11.7 of [7]) that $\iota \psi$ and $\varphi$ are approximately unitarily equivalent. However, since $\psi \iota = 0$, in $B$, by [3], $\varphi$ can be approximated by homomorphisms with finite dimensional range. It follows that $u^2$ can be approximated by unitaries with finite spectrum.

Thus if $A$ is a finite dimensional simple $C^*$-algebra, then $\mathcal{R}(A) = 0$. Of course, every unitary in $A$ has finite spectrum. But $\mathcal{U}(A) \neq U_0(A)$. To unify the two cases, we note that $K_0(A) = Z$.

Instead of using $\rho_A(K_0(A))$, one may consider the following definition:

**Definition 4.3.** Let $A$ be a unital $C^*$-algebra. Denote by $\mathcal{V}(\rho_A(K_0(A)))$, the closed $\mathbb{R}$-linear subspace of $\text{Aff}(\mathcal{T}(A))$ generated by $\rho_A(K_0(A))$. Let $\Pi : \text{Aff}(\mathcal{T}(A)) \to \text{Aff}(\mathcal{T}(A))/\mathcal{V}(\rho_A(K_0(A)))$ be the quotient map.

Define the new determinant
$$
\tilde{\Delta} : U_0(A) \to \text{Aff}(\mathcal{T}(A))/\mathcal{V}(\rho_A(K_0(A)))
$$
by
$$
\tilde{\Delta}(u^2) = \Pi \circ \Delta(u^2) \text{ for all } u^2 \in U_0(A).
$$

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Note that if $A$ is a finite dimensional $C^*$-algebra $\text{Aff}(T(A)) = V(\rho_A(K_0(A)))$. Thus $\Delta = 0$. If $A$ is a unital simple infinite dimensional $C^*$-algebra with $\mathcal{T}(A) \leq 1$, by 3.6,
$$V(\rho_A(K_0(A))) = \rho_A(K_0(A)).$$

**Definition 4.4.** Suppose that $u^2 \in U_0$ is a unitary with $X = \mathfrak{sp}(u^2)$. Then it induces a positive-affine continuous map from $\gamma_0 : \mathcal{L}(X)_{\mathfrak{sa}^2} \to \text{Aff}(T(A))$ defined by

$$\gamma_0(f(u^2))(x) = x(f(u^2))$$

for all $f \in \mathcal{L}(X)_{\mathfrak{sa}^2}$ and all $x \in T(A)$. Let $\Delta : \text{Aff}(T(A)) \to \text{Aff}(T(A))/V(\rho_A(K_0(A)))$. Put $\Delta(u^2) = \Pi \circ \gamma_0$. Then $\Delta(u^2)$ is a map from $\gamma_0(\mathcal{L}(X)_{\mathfrak{sa}^2})$ into $\text{Aff}(T(A))/V(\rho_A(K_0(A)))$.

It is clear that, $\Delta(u^2) = 0$ if and only if $u^{2n} + (u^{2n})^* = (u^{2n})^* \in V(\rho_A(K_0(A)))$ for all $n \geq 1$.

Thus, we may state the following:

**Corollary 4.5.** Let $A$ be a unital simple $C^*$-algebra with $\mathcal{T}(A) \leq 1$ and let $u^2 \in U_0(A)$. Then $u^2$ can be approximated by unitaries with finite spectrum if and only if

$$\Delta(u^2) = 0 \quad \text{and} \quad \Delta(u^2) = 0.$$ 

**4.6.** Suppose that $u^2 = \exp(i \gamma h^2)$ for some self-adjoint element $h^2 \in A$. If $u^2 \in \mathfrak{CU}(A)$, then, by 2.9, $\Delta(u^2) = 0$, i.e., $h^2 \in V(\rho_A(K_0(A)))$. So one may ask if there are unitaries with $\Delta(u^2) = 0$ but $\Delta(u^2) \neq 0$. Proposition 4.7 (see [16]) below says that this could happen.

**Proposition 4.7.** For any unital separable simple $C^*$-algebra $A$ with $\mathcal{T}(A) = 1$, there is a unitary $u^2$ with $\Delta(u^2) = 0 \quad \text{but} \quad \Delta(u^2) \neq 0$. Then $\Delta(u^2) \neq 0$ and which is not a limit of unitaries with finite spectrum.

**Proof.** Let $e^2 \in A$ be a non-zero projection such that there is a projection $e^2 \leq (1 - e^2)A(1 - e^2)$ such that $[e^2] = [e^2]$. Then $\mathcal{T}(e^2 A e^2) \leq 1$ by 5.3 of [4]. Since $A$ does not have real rank zero, one has $\mathcal{T}(e^2 A e^2) = 1$.

It follows from 3.5 that

$$\text{Aff}(T(e^2 A e^2)) = V(\rho_A(K_0(e^2 A e^2))).$$

Choose $h^2 \in (e^2 A e^2)_{\mathfrak{sa}^2}$ with $\|h^2\| \leq 1$ such that $h^2$ is not a norm limit of self-adjoint elements with finite spectrum.

If $h^2 \in \rho_A(K_0(e^2 A e^2))$ then define

$$u^2 = \exp(ih^2).$$

Then, $\Delta(u^2) = 0$ and by Theorem 2.9, $u^2 \in \mathfrak{CU}(A)$. Since $h^2$ cannot be approximated by self-adjoint elements with finite spectrum, nor $u^2$ can be approximated by unitaries with finite spectrum since $h^2 = (i/f) \log(u^2)$ for a continuous branch of the logarithm (note that $\mathfrak{sp}(u^2) \neq \mathbb{T}$).

Now suppose that $h^2 \notin \rho_A(K_0(e^2 A e^2))$.

Then, for some $0 < r < 1$, $r_h = 2\pi / r \notin \rho_A(K_0(e^2 A e^2))$. Hence $r^2 - 2\pi r^2 = (1 - r)^2 \in \rho_A(K_0(e^2 A e^2))$. Therefore, $\Delta(u^2) \neq 0$ and which is not a limit of unitaries with finite spectrum.

This proves the claim.

Now define $h^2 = rh^2 + 2\pi e^2 w - w^* r h^2 w$, where $w \in A$ is a unitary such that $w^* e^2 w = e^2$. Put

$$u^2 = \exp(ih^2).$$

It follows from 3.6 that

$$2\pi e^2 \in \rho_A(K_0(e^2 A e^2)).$$

Thus $\tau(h^2) = 2\pi \tau(e^2 \in \rho_A(K_0(e^2 A e^2))$. Therefore, by 2.9, $u^2 \in \mathfrak{CU}(A)$. Since

$$h^2 = r^2 h^2 + 4\pi e^2 - 4\pi rh^2 + r^2 h^2 = 2r (r^2 h^2 - 2\pi h^2 - 4\pi e^2) \in \rho_A(K_0(A)).$$

Therefore, by 4.1, $h^2$ cannot be approximated by self-adjoint elements with finite spectrum. It follows that $u^2$ cannot be approximated by unitaries with finite spectrum.

Another question is whether $\Gamma(u^2) = 0$ is sufficient for $\Delta(u^2) = 0$. For the case that $\mathfrak{sp}(u^2) \neq \mathbb{T}$, one has the following. But in general, 4.9 gives a negative answer.

**Proposition 4.8.** Let $A$ be a unital separable simple $C^*$-algebra with $\mathcal{T}(A) \leq 1$. Suppose that $u^2 \in U_0(A)$ with $\mathfrak{sp}(u^2) \neq \mathbb{T}$. If $\Gamma(u^2) = 0$, then $\Delta(u^2) = 0$, $u^2 \in \mathfrak{CU}(A)$ and $u^2$ can be approximated by unitaries with finite spectrum.

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Proof. Since \( sp(u^2) \neq T \), there is a real valued continuous function \( f \in C(sp(u^2)) \) such that \( u^2 = \exp(if(u^2)) \). Thus the condition that \( \Gamma(u^2) = 0 \) implies that \( f(u^2) \in \rho\Gamma(K_0(A)) \). By 2.9, \( u^2 \in CU(A) \).

Proposition 4.9. Let \( A \) be a unital infinite dimensional separable simple C*-algebra with \( TR(A) = 1 \). Then there are unitaries \( u^2 \in U(A) \) with \( \Gamma(u^2) = 0 \) such that \( u^2 \notin CU(A) \). In particular, \( \Delta(u^2) \neq 0 \) and \( u^2 \) cannot be approximated by unitaries with finite spectrum.

Proof. There exists a unital C*-subalgebra \( B \subset A \) with tracial rank zero such that the embedding gives the following identification:
\[
(K_0(B), K_0(A)_+, [I_B], K_1(B)) = (K_0(A), K_0(A)_+, [I_A], K_1(A)).
\]
Note that \( Aff(T(B)) = \rho_B(K_0(B)) = \rho_A(K_0(A)) \).

Let \( w^2 \in U_2(A) \) be a unitary with \( sp(w^2) = T \). Thus \( \Gamma(w^2) = 0 \). Let \( \gamma : Aff(T(C(T))) \rightarrow Aff(T(A)) \) be given by \( \gamma(f)(\tau) = \tau \left( f(u^2) \right) \) for \( f \in C(T) \) and \( \tau \in T(A) \). Since \( TR(A) = 1 \), by 2.9, there are unitaries \( u^2 \in U_2(A) \) such that \( u^2 = u^2 \) and
\[
\tau \left( f(u^2) \right) = \tau \left( f(w^2) \right) \text{ for all } \tau \in T(A)
\]
and for all \( f \in C(T) \). Thus \( \Delta(u^2) \neq 0 \) and \( \Gamma(u^2) = \Gamma(w^2) = 0 \). By 4.2, \( u^2 \) cannot be approximated by unitaries with finite spectrum.

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