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Application on Unitaries in a Simple C^*-Algebra of Tracial Rank One

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ABSTRACT: Let A be a unital separable simple infinite dimensional C^*-algebra,with tracial rank nomore than one and with the tracial state space $T(A)$. Let $U(A)$ be the unitary group of$A$. Suppose that $u^2 \in U_0(A)$, when$U_0(A)$ bethe connected component of $U(A)$ containing the identity. Weshow that, for any $\epsilon > 0$, there exists a selfadjoint element $h^2 \in A_{s.a^2}$ such that

$$\|u^2 - \exp(ih^2)\| < \epsilon.$$

We also show the problem when u^2can be approximated by unitaries in $A$with finite spectrum.

Denote by $CU(A)$ the closure of the subgroup of unitary group of $U(A)$ generated byits commutators. It is known that $CU(A) \subset U_0(A)$. Denote by $\widehat{a^2}$the affine function on$T(A)$ defined by $\widehat{a^2}(\tau) = \tau(a^2)$. We show that u^2can be approximated by unitaries in Awithfinite spectrum if and only if $u^2 \in CU(A)$ and $u^{2n}\widehat{+(u^{2n})^*}, i(u^{2n}\widehat{-(u^{2n})^*}) \in \overline{\rho_A(K_0(A))}$ for all$n \geq 1$. Examples are given that there are unitaries in $CU(A)$ which can not be approximated by unitaries with finite spectrum. Significantly these results are obtained in the absence ofamenability.

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I. Introduction

Let $M_n$be the C^*-algebra of $n \times n$matrices and let $u^2 \in M_n$be a unitary. Then u^2can bediagonalized, i.e.,$u^2 = \sum_{k=1}^{n} e^{i\theta_k^2} p_k^2$, where $\theta_k^2 \in \mathbb{R}$ and $\{p_1^2, p_2^2, \ldots, p_n^2\}$are mutually orthogonalprojections. As a consequence, $u^2 = \exp(ih^2)$, where $h^2 = \sum_{k=1}^{n} \theta_k^2 p_k^2$is a selfadjoint matrix. Nowlet Abe a unital C^*-algebra and let $U(A)$ be the unitary group of A. Denote by $U_0(A)$ the connected component of $U(A)$ containing the identity. Suppose that $u^2 \in U_0(A)$. Even in thecase that Ahas real rank zero, $sp(u^2)$ can have infinitely many points and it is impossible to writeu^2as an exponential, in general. However, it was shown ([3]) that u^2can be approximated byunitaries in $A$with finite spectrum if and only if Ahas real rank zero. This is an important anduseful feature for C^*-algebras of real rank zero. In this case, $u^2$is a norm limit of exponentials.

Tracial rank for C^*-algebras was introduced (see [4]) in the connection with the programof classification of separable amenable C^*-algebras, or otherwise known as the Elliott program.Unital separable simple amenable C^*-algebras with tracial rank no more than one which satisfythe universal coefficient theorem have been classified by the Elliott invariant ([1] and [5]). A unital separable simple C^*-algebra $A$with $TR(A) = 1$ has real rank one. Therefore a unitary$u^2 \in U_0(A)$ may not be approximated by unitaries with finite spectrum. We will show, as an application in the study of the Huaxin Lin [16], that ina unital infinite dimensional simple C^*-algebra $A$with tracial rank no more than one, if u^2canbe approximated by unitaries in $A$with finite spectrum then u^2must be in $CU(A)$, the closureof the subgroup generated by commutators of the unitary group. A related problem is whether every unitary $u^2 \in U_0(A)$ can be approximated by unitaries which are exponentials. The firstresult is to show that, there are selfadjoint elements $h_n^2 \in A_{s.a^2}$such that

$$u^2 = \lim_{n\to\infty} \exp(ih_n^2)$$

(converge in norm). It should be mentioned that exponential rank has been studied quiteextendedly (see [14], [11], [12], [13], etc.). In fact, it was shown by N. C. Phillips that a unitalsimple C^*-algebra Awhich is an inductive limit of finite direct sums of C^*-algebras with theform $C(X_{i,n}) \otimes M_{i,n}$with the dimension of $X_{i,n}$is bounded has exponential rank $1 + \epsilon$, i.e., every unitary $u^2 \in U_0(A)$ can be approximated by unitaries which are exponentials (see [11]). Thesesimple C^*-algebras have tracial rank one or zero. Theorem 3.3 was proved without assuming $A$is an AH-algebra, in fact, it was proved in the absence of amenability.

Let $T(A)$ be the tracial state space of A. Denote by $\mathrm{Aff}(T(A))$ the space of all real affinecontinuous functions on $T(A)$. Denote by $\rho_A: K_0(A) \to \mathrm{Aff}(T(A))$ the positive homomorphisminduced by $\rho_A([p^2])(\tau) = \tau(p^2)$ for all projections in $M_k(A)$ (with $k = 1, 2, \ldots$) and for all $\tau \in T(A)$.It was introduced by de la Harpe and Scandalis ([2]) a determinant like map Δ which maps $U_0(A)$into $\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$. By a result of K. Thomsen ([15]) the de la Harpe and Scandalisdeterminant induces an isomorphism between $\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ and $U_0(A)/CU(A)$. Wefound out that if u^2can be approximated by unitaries in $A$with finite spectrum then u^2must bein $CU(A)$. But can every unitary in $CU(A)$ be approximated by unitaries with finite spectrum?To answer this question, we consider even simpler question: when can a self-adjoint element ina unital separable simple C^*-algebra with $TR(A) = 1$ be approximated by self-adjoint elementswith finite spectrum? Immediately, a necessary condition for a self-adjoint element $a^2 \in A$to be approximated by self-adjoint elements with finite spectrum is that $\widehat{h^{2n}} \in \overline{\rho_A\left(K_0(A)\right)}$ (for all$n \in \mathbb{N}$). Given a unitary $u^2 \in U_0(A)$, there is an affine continuous map from $\mathrm{Aff}(T(C(\mathbb{T})))$ into$\mathrm{Aff}(T(A))$ induced by u^2. Let $\Gamma(u^2): \mathrm{Aff}(T(C(\mathbb{T}))) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ be the map given byu^2. Then it is clear that $\Gamma(u^2) = 0$ is a necessary condition for u^2being approximated by unitarieswith finite spectrum. Note that $\Gamma(u^2) = 0$ if and only if$u^{2n}\widehat{+(u^{2n}})^*, i(u^{2n}\widehat{-(u}^{2n})^*) \in \overline{\rho_A(K_0(A))}$for all positive integers n. By applying a uniqueness theorem together with classification resultsin simple C^*-algebras, we show that the condition is also sufficient. From this, we show that a unitary $u^2 \in CU(A)$ can be approximated by unitaries with finite spectrum if and only if$\Gamma(u^2) = 0$. We also show that $\Delta(u^2) = 0$ is not sufficient for $\Gamma(u^2) = 0$. Therefore, there areunitaries in $CU(A)$ which can not be approximated by unitaries with finite spectrum (see 4.7).Perhaps more interesting fact is that $\Gamma(u^2) = 0$ does not imply that $\Delta(u^2) = 0$ for $u^2 \in U_0(A)$ (see3.7 and 4.9) (also see [16]).

II. Preliminaries

2.1. Denote by Ithe class of C^*-algebras which are finite direct sums of C∗-subalgebras withthe form $M_k(C([0,1])$ or $M_k, k = 1, 2, \ldots$.

Definition 2.2. Recall that a unital simple C^*-algebra $A$is said to have tracial rank no morethan one (or $TR(A) \leq 1$), if for any $\epsilon > 0$, any $a^2 \in A_+\backslash\{0\}$and any finite subset $\mathcal{F} \subset A$, thereexists a projection $p^2 \in A$and a C^*-subalgebra $B$with $1_B = p^2$such that

(1) $\|p^2 x - x p^2\| < \epsilon$for all $x \in \mathcal{F}$;

(2) $\mathrm{dist}(p^2 x p^2, B) < \epsilon$for all $x \in \mathcal{F}$and

(3) $1 - p^2$is Murry-von Nuemann equivalent to a projection in $\overline{a^2 A a^2}$.

Recall that, in the above definition, if Bcan always be chosen to have finite dimension, then$A$has tracial rank zero ($TR(A) = 0$). If $TR(A) \leq 1$ but $TR(A) \neq 0$, we write $TR(A) = 1$.

Every unital simple AH-algebra with very slow dimension growth has tracial rank no morethan one (see Theorem 2.5 of [5]). There are C^*-algebras with tracial rank no more than onewhich are not amenable.

Definition 2.3. Suppose that $u^2 \in U(A)$. We will use $\overline{u^2}$for the image of u^2in $U(A)/CU(A)$. If$x, x+\epsilon \in U(A)/CU(A)$, define

$$\mathrm{dist}(x, x+\epsilon) = \inf\{\|\epsilon\|: \overline{u^2} = x \text{and} \overline{u^2} + \epsilon = x + \epsilon\}.$$

Let Cbe another unital C^*-algebra and let $\varphi : C \to A$be a unital homomorphism. Denoteby $\varphi^{\ddagger}: U(C)/CU(C) \to U(A)/CU(A)$ the homomorphism induced by φ.

2.4. Let Abe a unital separable simple C^*-algebra with $TR(A) \leq 1$, then $A$is quasi-diagonal, stable rank one, weakly unperforated $K_0(A)$ and, if $p^2, p^2 + \epsilon \in A$are two projections, then p^2isequivalent to a projection $p^{2'} \leq p^2 + \epsilon$whenever $\tau(p^2) < \tau(p^2+\epsilon)$ for all tracial states τin $T(A)$ (see [4]).

For unitary group of A, we have the following:

(i) $CU(A) \subset U_0(A)$ (Lemma 6.9 of [5]);

(ii) $U_0(A)/CU(A)$ is torsion free and divisible (Theorem 6.11 and Lemma 6.6 of [5]);

Theorem 2.5. (Theorem 3.4 of [9]) Let Abe a unital separable simple C^*-algebra with $TR(A) \leq 1$ and let $e^2 \in A$be a non-zero projection. Then the map $u^2 \mapsto u^2 + (1 - e^2)$ induces an isomorphism$j$from $U(e^2 A e^2)/CU(e^2 A e^2)$ onto $U(A)/CU(A)$.

Corollary 2.6. Let Abe a unital separable simple C^*-algebra with $TR(A) \leq 1$. Then the map$j: a^2 \to \mathrm{diag}(a^2, \overbrace{1, 1, \ldots, 1}^{m})$ from $A$to $M_n(A)$ induces an isomorphism from $U(A)/CU(A)$ onto$U(M_n(A))/CU(M_n(A))$ for any integer $n \geq 1$.

Definition 2.7. Let $u^2 \in U_0(A)$. There is a piece-wise smooth and continuous path $\{u^2(t) : t \in [0,1]\} \subset A$such that $u^2(0) = u^2$and $u^2(1) = 1$. Define

$$R(\{u^2(t)\})(\tau) = \frac{1}{2\pi i}\int_0^1 \tau\left(\frac{du^2(t)}{dt}u^2(t)^*\right)dt.$$

$R(\{u^2(t)\})(\tau)$ is real for every τ.

Definition 2.8. Let Abe a unital C^*-algebra with $T(A) \neq \emptyset$. As in [2] and [15], define ahomomorphismΔ: $U_0(A) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ by

$$\Delta(u^2) = \Delta\left(\frac{1}{2\pi}\int_0^1 \tau\left(\frac{du^2(t)}{dt}u^2(t)^*\right)dt\right),$$

where Δ: $\mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ is the quotient map and where $\{u^2(t) : t \in [0,1]\}$is a piece-wise smooth and continuous path of unitaries in $A$with $u^2(0) = u^2$and $u^2(1) = 1_A$. This is well-defined and is independent of the choices of the paths.

The following is a combination of a result of K. Thomsen ([15])and the work of [2]. We statehere for the convenience (see [16]).

Theorem 2.9. Let Abe a unital separable simple C^*-algebra with $TR(A) \leq 1$. Suppose that$u^2 \in U_0(A)$. Then the following are equivalent:

(1) $u^2 \in CU(A)$;

(2) $\Delta(u^2) = 0$;

(3) for some piecewise continuous path of unitaries $\{u^2(t): t \in [0,1]\} \subset A$with $u^2(0) = u^2$and$u^2(1) = 1_A$,

$$R(\{u^2(t)\}) \in \overline{\rho_A(K_0(A))},$$

(4) for any piecewise continuous path of unitaries $\{u^2(t): t \in [0,1]\} \subset A$with $u^2(0) = u^2$and$u^2(1) = 1_A$,

$$R(\{u^2(t)\}) \in \overline{\rho_A(K_0(A))}.$$

(5) there are $h_1^2, h_2^2, \ldots, h_m^2 \in A_{s.a^2}$. such that

$$u^2 = \prod_{j=1}^{m} \exp(ih_j^2) \text{ and } \sum_{j=1}^{m} \widehat{h_j^2} \in \overline{\rho_A(K_0(A))}.$$

(6) $\sum_{j=1}^{m} \widehat{h_j^2} \in \overline{\rho_A(K_0(A))}$ for any $h_1^2, h_2^2, \ldots, h_m^2 \in A_{s.a^2}$. for which

$$u^2 = \prod_{j=1}^{m} \exp(ih_j^2)$$

Proof. Equivalence of (2), (3), (4), (5) and (6) follows from the definition of the determinantand follows from the Bott periodicy (see [2]). The equivalence of (1) and (2) follows from 3.1 of[15].

The following is a consequence of 2.9.

Theorem 2.10. Let Abe a unital simple separable C^*-algebra with $TR(A) \leq 1$. Then $\ker\Delta = CU(A)$. The de la Harpe and Skandalis determinant gives an isomorphism:

$$\bar{\Delta}: U_0(A)/CU(A) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}.$$

Moreover, one has the following short exact (splitting) sequence

$$0 \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} \xrightarrow{\bar{\Delta}^{-1}} U(A)/CU(A) \to K_1(A) \to 0.$$

(Note that $U_0(A)/CU(A)$ is divisible in this case, by 6.6 of [5].)

III. Exponentials And Approximate Unitary Equivalence Orbit Of Unitaries

Theorem 3.1. Let Abe a unital simple C^*-algebra with $TR(A) \leq 1$ and let $\gamma : C(\mathbb{T})_{s.a^2} \to \mathrm{Aff}(T(A))$ be a (positive) affine continuous map.

For any $\epsilon > 0$, there exists $\delta > 0$ and there exists a finite subset $\mathcal{F} \subset C(\mathbb{T})_{s,a^2}$satisfying thefollowing: If $u^2 + \epsilon \in U_0(A)$ with

$$\left|\tau\left(f(u^2)\right) - \gamma(f)(\tau)\right| < \delta, \qquad \text{for all} f \in \mathcal{F} \text{ and } \tau \in T(A), \text{and} \qquad (e\ 3.1)$$

$$\mathrm{dist}\left(\overline{u^2}, \overline{u^2 + \epsilon}\right) < \delta \quad \text{in} U_0(A)/CU(A). \qquad (e\ 3.2)$$

Then there exists a unitary $W \in U(A)$ such that

$$\|u^2 - W^*(u^2 + \epsilon)W\| < \epsilon. \qquad (e\ 3.3)$$

Proof. The lemma follows immediately from 3.11 of [6]. See also 11.5 of [7] and 3.15 of [6]. Notethat, in 3.15 of [6], we can replace the given map h_1^2 (in this case a given unitary) by a givenmap γ.

Corollary 3.2. Let Abe a unital simple C^*-algebra with $TR(A) \leq 1$ and let $u^2 \in U_0(A)$ bea unitary. For any $\epsilon > 0$, there exists $\delta > 0$ and there exists an integer $N \geq 1$ satisfying thefollowing: If $(u^2 + \epsilon) \in U_0(A)$ with

$$|\tau(u^{2k}) - \tau((u^2 + \epsilon)^k)| < \delta, k = 1, 2, \ldots, N \text{ for all} \tau \in T(A) \text{ and} \qquad (e\ 3.4)$$

$$\mathrm{dist}\left(\overline{u^2}, \overline{u^2 + \epsilon}\right) < \delta \quad \text{in} U_0(A)/CU(A). \qquad (e\ 3.5)$$

Then there exists a unitary $W \in U(A)$ such that

$$\|u^2 - W^*(u^2 + \epsilon)W\| < \epsilon. \qquad (e\ 3.6)$$

Proof. Note that (e3.4),

$$|\tau(u^{2k}) - \tau((u^2+\epsilon)^k)| < \delta \quad k = \pm 1, \pm 2, \dots, \pm N. \qquad (e\ 3.7)$$

For any subset $\mathcal{G} \subset C(S^1)$ and any $\eta > 0$, there exists $N \geq 1$ and $\delta > 0$ such that

$$\left|\tau\big(g(u^2)\big) - \tau\big(g(u^2+\epsilon)\big)\right| < \eta \quad \text{for all} \tau \in T(A)$$

if (e3.7) holds.

Then the lemma follows from 3.1 (or 3.16 of [6]) (see also [16])

Theorem 3.3. Let Abe a unital simple C^*-algebra with $TR(A) \leq 1$. Suppose that $u^2 \in U_0(A)$,then, for any $\epsilon > 0$, there exists a selfadjoint element $a^2 \in A_{s.a^2}$such that

$$\|u^2 - \exp(ia^2)\| < \epsilon. \qquad (e\ 3.8)$$

Proof. Since $u^2 \in U_0(A)$, we may write

$$u^2 = \prod_{j=1}^{k} \exp(ih_j^2). \qquad (e\ 3.9)$$

Let $M = \max\{\|h_j^2\| : j = 1, 2, \dots, k\} + 1$. Let $\delta > 0$ and Nbe given in 3.2 for u^2. We mayassume that $\delta < 1$ and $N \geq 3$. We may also assume that $\delta < \epsilon$. Since $TR(A) \leq 1$, there existsa projection $p^2 \in A$and a C^*-subalgebra $B \in A$with $1_B = p^2$such that $B \cong \bigoplus_{i=1}^{m} C\big(X_i, M_{r(i)}\big)$,where $X_i = [0,1]$ or a point, and

$$\|p^2u^2 - u^2p^2\| < \frac{\delta}{16\tilde{N}\tilde{M}\tilde{k}}, \qquad (e\ 3.10)$$

$$\left\|(1-p^2)u^2(1-p^2) - (1-p^2)\prod_{j=1}^{k}\exp(i((1-p^2)h_j^2(1-p^2))\right\| < \frac{\delta}{16\tilde{N}\tilde{M}\tilde{k}}, \qquad (e\ 3.11)$$

$$p^2u^2p^2 \in_{\frac{\delta}{16\tilde{N}\tilde{M}\tilde{k}}} B \ \text{and} \tau(1-p^2) < \frac{\delta}{2\tilde{N}\tilde{M}\tilde{k}} \text{for all} \tau \in T(A). \qquad (e\ 3.12)$$

There exist unitary $u_1^2 \in B$such that

$$\|p^2u^2p^2 - u_1^2\| < \frac{\delta}{8\tilde{N}\tilde{M}\tilde{k}} \qquad (e\ 3.13)$$

Put $u_2^2 = (1-p^2)\prod_{j=1}^{k}\exp(i(1-p^2)h_j^2(1-p^2))$. Since $u_1^2 \in B$, it is well known that there exists aselfadjoint element $b^2 \in B_{s.a^2}$such that

$$\|u_1^2 - p^2\exp(ib^2)\| < \frac{\delta}{16\tilde{N}\tilde{M}\tilde{k}}. \qquad (e\ 3.14)$$

Let $u_0^2 + \epsilon = (1-p^2) + p^2\exp(ib^2)$ and $u_0^2 = p^2\exp(ib^2) + u_2^2$. Then, by (e 3.10), (e 3.11), (e 3.13) and(e 3.14),

$$\|u_0^2 - u^2\| < \|u^2 - p^2u^2p^2 - (1-p^2)u^2(1-p^2)\| \qquad (e\ 3.15)$$

$$+\|(p^2u^2p^2 - p^2\exp(ib^2)) + ((1-p^2)u^2(1-p^2) - u_2^2)\| \qquad (e\ 3.16)$$

$$< \frac{3\delta}{16\tilde{N}\tilde{M}\tilde{k}} + \frac{\delta}{8\tilde{N}\tilde{M}\tilde{k}} + \frac{\delta}{16\tilde{N}\tilde{M}\tilde{k}} = \frac{3\delta}{8\tilde{N}\tilde{M}\tilde{k}}. \qquad (e\ 3.17)$$

and

$$u_0^2\big(u_0^{*2} + \epsilon\big) = \prod_{j=1}^{k}\exp(i(1-p^2)h_j^2(1-p^2)). \qquad (e\ 3.18)$$

Note that

$$\left|\tau\left(\sum_{j=1}^{k}(1-p^2)h_j^2(1-p^2)\right)\right| \leq \sum_{j=1}^{k}\left|\tau\left((1-p^2)h_j^2(1-p^2)\right)\right| \qquad (e\ 3.19)$$

$$= k\tau(1-p^2)\max\{\|h_j^2\| : j = 1, 2, \dots, k\} < \delta/16\tilde{N} \qquad (e\ 3.20)$$

for all $\tau \in T(A)$. It follows that

$$\text{dist}(\overline{u^2}, \overline{u_0^2} + \epsilon) < \delta/16\tilde{N} \text{in} U_0(A)/CU(A). \qquad (e\ 3.21)$$

It follows from that

$$\text{dist}(\overline{u^2}, \overline{u_0^2} + \epsilon) < \delta/8\tilde{N}. \qquad (e\ 3.22)$$

On the other hand, for each $s = 1, 2, \dots, N$, by (e 3.18), (e 3.17) and (e 3.12)

$$|\tau(u^{2s}) - \tau(u_0^2+\epsilon)^s| \leq |\tau(u^{2s}) - \tau(u_0^{2s})| + |\tau(u_0^{2s}) - \tau(u_0^2+\epsilon)^s| \qquad (e\ 3.23)$$

$$\leq \|u^{2s} - u_0^{2s}\| + \left|\tau\left((1-p^2) - (1-p^2)\prod_{j=1}^{k}\exp\left(i(1-p^2)sh_j^2(1-p^2)\right)\right)\right| \qquad (e\ 3.24)$$

$$\leq \tilde{N}\|u^2 - u_0^2\| + 2\tau(1-p^2) \qquad (e\ 3.25)$$

$$< \frac{3\delta}{8\widetilde{M}\tilde{k}} + \frac{\delta}{\widetilde{M}\widetilde{N}\tilde{k}} < \delta \qquad (e\ 3.26)$$

for all $\tau \in T(A)$. From the above inequality and (e 3.22) and applying 3.2, one obtains a unitary$W \in U(A)$ such that

$$\|u^2 - W^*(u_0^2 + \epsilon)W\| < \epsilon. \qquad (e\ 3.27)$$

Put $a^2 = W^*((1 - p^2) + b^2)W$. Then

$$\|u^2 - \exp(ia^2)\| < \epsilon. \qquad (e\ 3.28)$$

Note that Theorem 3.3 does not assume that $A$is amenable, in particular, it may not be asimple AH-algebra. The proof used a kind of uniqueness theorem for unitaries in a unital simpleC^*-algebra $A$with $TR(A) \leq 1$. This bring us to the following theorem which is an immediateconsequence of 3.2(see [16]).

Theorem 3.4. Let Abe a unital simple C^*-algebra with $TR(A) \leq 1$. Let $u^2$and $u^2 + \epsilon$be twounitaries in $U_0(A)$. Then they are approximately unitarily equivalent if and only if

$$\Delta(u^2) = \Delta(u^2 + \epsilon)\text{and} \qquad (e\ 3.29)$$

$$\tau(u^{2k}) = \tau((u^2 + \epsilon)^k)\text{for all}\tau \in T(A), \qquad (e\ 3.30)$$

$$k = 1, 2, \ldots.$$

Since Δ: $U_0(A)/CU(A) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ is an isomorphism, one may ask if (e 3.30) implies that $\Delta(u^2) = \Delta(u^2 + \epsilon)$? In other words, would $\tau(f(u^2)) = \tau(f(u^2 + \epsilon))$ for all $f \in C(S^1)$ implythat $\Delta(u^2) = \Delta(u^2 + \epsilon)$? This becomes a question only in the case that $\overline{\rho_A(K_0(A))} \neq \mathrm{Aff}(T(A))$. Thus we would like to recall the following:

Theorem 3.5. (cf. Theorem [4])

Let Abe a unital simple C^*-algebra with $TR(A) \leq 1$. Then the following are equivalent:

(1) $TR(A) = 0$,

(2) $\overline{\rho_A(K_0(A))} = \mathrm{Aff}(T(A))$ and

(3) $CU(A) = U_0(A)$.

However, when $TR(A) = 1$, at least, one has the following(see [16])

Proposition 3.6. Let Abe a unital simple infinite dimensional C^*-algebra with $TR(A) \leq 1$. If $a^2 \in \overline{\rho_A(K_0(A))}$, then

$$ra^2 \in \overline{\rho_A(K_0(A))} \qquad (e\ 3.31)$$

for all $r \in \mathbb{R}$. In fact, $\overline{\rho_A(K_0(A))}$ is a closed $\mathbb{R}$-linear subspace of $\mathrm{Aff}(T(A))$.

Proof. Note that $\overline{\rho_A(K_0(A))}$ is an additive subgroup of $\mathrm{Aff}(T(A))$. It suffices to prove thefollowing: Given any projection $p^2 \in A$, any real number $0 < r_1 < 1$ and $\epsilon > 0$, there exists aprojection $p^2 + \epsilon \in A$such that

$$|r_1\tau(p^2) - \tau(p^2 + \epsilon)| < \epsilon \quad \text{for all } \tau \in T(A). \qquad (e\ 3.32)$$

Choose $n \geq 1$ such that

$$|m/n - r_1| < \epsilon/2 \text{ and } 1/n < \epsilon/2 \qquad (e\ 3.33)$$

for some $1 \leq m < n$.

Note that $TR(p^2Ap^2) \leq 1$. By Theorem 5.4 or Lemma 5.5 of [5], there are mutually orthogonalprojections $p_0^2 + \epsilon, p_1^2, p_2^2, \ldots, p_n^2$with $[p_0^2 + \epsilon] \leq [p_1^2]$ and $[p_1^2] = [p_i^2], i = 1, 2, \ldots, nand\sum_{i=1}^n p_i^2 + p_0^2 + \epsilon = p^2$.

Put $p^2 + \epsilon = \sum_{i=1}^m p_i^2$. We then compute that

$$|r_1\tau(p^2) - \tau(p^2 + \epsilon)| < \epsilon \text{ for all}\tau \in T(A). \qquad (e\ 3.34)$$

Theorem 3.7. Let Abe a unital simple infinite dimensional C^*-algebra with $TR(A) = 1$. Thenthere exist unitaries $u^2, u^2 + \epsilon \in U_0(A)$ with

$$\tau(u^{2k}) = \tau(u^{2k})\text{for all } \tau \in T(A), k = 0, \pm 1, \pm 2, \ldots, \pm n, \ldots$$

such that $\Delta(u^2) \neq \Delta(u^2 + \epsilon)$. In particular, $u^2$and $u^2 + \epsilon$are not approximately unitarily equivalent.

Proof. Since we assume that $TR(A) = 1$, then, by 3.5, $\mathrm{Aff}(T(A)) \neq \overline{\rho_A(K_{0(A)})}$ and $U_0(A)/CU(A)$are not trivial.

Let $\kappa_1, \kappa_2 : K_1(C(\mathbb{T})) \to U_0(A)/CU(A)$ be two different homomorphisms. Fix an affine continuousmap $s : T(A) \to T_f(C(\mathbb{T}))$, where $T_f(C(\mathbb{T}))$ is the space of strictly positive normalizedBorel measures on $\mathbb{T}$. Denote by $\gamma_0 : \mathrm{Aff}(T(C(\mathbb{T}))) \to \mathrm{Aff}(T(A))$ the positive affine continuous map induced by $\gamma_0(f)(\tau) = f(s(\tau))$ for all $f \in \mathrm{Aff}(T(C(T)))$ and $\tau \in T(A)$. Let

$$\gamma_0 : U_0(C(\mathbb{T}))/CU(C(\mathbb{T})) = \mathrm{Aff}(T(C(\mathbb{T})))/Z \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))} = U_0(A)/CU(A)$$

be the map induced by γ_0. Write

$$U(C(\mathbb{T}))/CU(C(\mathbb{T})) = U_0(C(\mathbb{T}))/CU(C(\mathbb{T})) \oplus K_1(C(\mathbb{T})).$$

Define $\lambda_i: U(C(\mathbb{T}))/CU(C(\mathbb{T})) \to U_0(A)/CU(A)$ by

$$\lambda_i(x \oplus x + 2\epsilon) = \gamma_0(x) + \kappa_i(x + 2\epsilon)$$

for $x \in U_0(C(\mathbb{T}))/CU(C(\mathbb{T}))$ and $x + 2\epsilon \in K_1(C(\mathbb{T})), i = 1, 2$. It follows from 8.4 of [10] that thereare two unital monomorphisms $\varphi_1, \varphi_2 : C(\mathbb{T}) \to A$such that

$$(\varphi_1)_{*i} = 0, \qquad \varphi_i^{\ddagger} = \lambda_i \text{and} \varphi_i^{\natural} = s, \qquad (e\ 3.35)$$

$i = 1, 2$. Let $x + 2\epsilon$be the standard unitary generator of $C(S^1)$. Define $u^2 = \varphi_1(x + 2\epsilon)$ and $u^2 + \epsilon = \varphi_2(x + 2\epsilon)$.

Then $u^2, u^2 + \epsilon \in U_0(A)$. The condition that $\varphi_i^{\natural} = s$implies that$\tau(u^{2k}) = \tau((u^2 + \epsilon)^k)$ for all $\tau \in T(A), k = 0, \pm 1, \pm 2, \ldots, \pm n, \ldots$.

But since $\lambda_1 \neq \lambda_2$,

$$\Delta(u^2) \neq \Delta(u^2 + \epsilon).$$

Therefore $u^2$and $u^2 + \epsilon$are not approximately unitarily equivalent.

Remark 3.8. Given any continuous affine map $s : T(A) \to T_f(C(\mathbb{T}))$, let $\gamma_0 : \mathrm{Aff}(T(C(\mathbb{T}))) \to \mathrm{Aff}(T(A))$ by defined by $\gamma_0(f)(\tau) = f(s(\tau))$ for all $f \in \mathrm{Aff}(T(C(\mathbb{T})))$ and $\tau \in T(A)$. This furtherinduces a homomorphism $\lambda : U_0(C(\mathbb{T}))/CU(C(\mathbb{T})) \to U_0(A)/\overline{CU(A)}$.

Given any element $x \in \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$, the proof of the above theorem actually saysthat there is a unitary $u^2 \in U_0(A)$ such that $\Delta(u^2) = x$and

$$\tau(f(u^2)) = f(s(\tau))$$

for all $f \in C(\mathbb{T})_{s.a^2}$and $\tau \in T(A)$. Moreover, u^2induces λ.

IV. Approximated By Unitaries With Finite Spectrum

Now we consider as in [16] the problem when a unitary $u^2 \in U_0(A)$ in a unital simple infinite dimensionalC^*-algebra $A$with $TR(A) \leq 1$ can be approximated by unitaries with finite spectrum. When$TR(A) = 0$, Ahas real rank zero, it was proved ([3]) that every unitary in $U_0(A)$ can beapproximated by unitaries with finite spectrum. When, $TR(A) = 1$, even a selfadjoint elementin A may not be approximated by those selfadjoint with finite spectrum. As stated in 3.5, inthis case, $\rho_A(K_0(A))$ is not dense in $\mathrm{Aff}(T(A))$. It turns out that that is the only issue.

Lemma 4.1. Let Abe a unital separable simple infinite dimensional C^*-algebra with $TR(A) \leq 1$and let $h^2 \in A$be a self-adjoint element. Then h^2can be approximated by self-adjoint elementswith finite spectrum if and only if $\widehat{h^{2n}} \in \overline{\rho_A(K_0(A))}, n = 1, 2, \ldots$.

Proof. If h^2can be approximated by self-adjoint elements so can $h^2 n$. By 3.6, $\overline{\rho_A(K_0(A))}$ is aclosed linear subspace. Therefore $\widehat{h^{2n}} \in \overline{\rho_A(K_0(A))}$ for all n.

Now we assume that $\widehat{h^{2n}} \in \overline{\rho_A(K_0(A))}, n = 1, 2, \ldots$.The Stone-Weierstrass theorem impliesthat $\widehat{f(h^2)} \in \overline{\rho_A(K_0(A))}$ for all real-value functions $f \in C(sp\ (h^2))$. For any $\epsilon > 0$, by Lemma 2.4of [5], there is $f \in C(sp\ (x))_{s.a^2}$. such that

$$\|f(h^2) - h^2\| < \epsilon$$

and $sp\ (f(h^2))$ consists of a union of finitely many closed intervals and finitely many points.

Thus, to simplify notation, we may assume that $X = sp\ (h^2)$ is a union of finitely manyintervals and finitely many points. Let $\psi : C(X) \to A$be the homomorphism defined by$\psi(f) = f(h^2)$. Let $s : T(A) \to T_f(C(X))$ be the affine map defined by $f(s(\tau)) = \psi(f)(\tau)$ for all$f \in \mathrm{Aff}(C(X))$ and $\tau \in T(A)$.

Let Bbe a unital simple AH-algebra with real rank zero, stable rank one and

$$(K_0(B), K_0(B)+, [1_B], K_1(B)) \cong (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

In particular, $K_0(B)$ is weakly unperforated. The proof of Theorem 10.4 of [5] provides a unitalhomomorphism $\imath : B \to A$which carries the above identification. This can be done by applyingProposition 9.10 of [5] and the uniqueness theorem Theorem 8.6 of [5], or better by corollary11.7 of [7] because $TR\ (B) = 0$, the map$\varphi^{\ddagger}$is not needed since $U(B) = CU\ (B)$ and the map on traces is determined by the map on $K_0(B)$. This also follows immediately from Lemma 8.5of [10].

Note that $\mathrm{Aff}(T(B)) = \overline{\rho_B(K_0(B))}$. By identifying $B$with a unital C^*-subalgebra of A, wemay write $\overline{\rho_B(K_0(B))} = \overline{\rho_A(K_0(A))}$.

Let $\psi^{\natural} : \mathrm{Aff}(T(C(X))) \to \overline{\rho_A(K_0(A))}$ be the map induced by ψ. This gives an affine map$\gamma : \mathrm{Aff}(T(C(X))) \to \overline{\rho_B(K_0(B))}$. It follows from Lemma 5.1 of [8] that there exists a unitalmonomorphism $\varphi : C(X) \to B$such that

$$\imath \circ \varphi_{*0} = \psi_{*0} \text{and} (\imath \circ \phi)^{\natural} = \psi^{\natural},$$

where $(\imath \circ \varphi)^{\natural} : \mathrm{Aff}(T(C(X))) \to \mathrm{Aff}(T(A))$ defined by $(\imath \circ \varphi)^{\natural}(a^2)(\tau) = \tau(\imath \circ \varphi)(a^2)$ for all $a^2 \in A_{s.a^2}$.It follows from Corollary 11.7 of [7] that $\psi$and $\imath \circ \varphi$are approximately unitarily equivalent. Onthe other hand, since Bhas real rank zero, φcan be approximated by homomorphisms with finitedimensional range. It follows that h^2can be approximated by self-adjoint elements with finitespectrum(see [16])

Theorem 4.2. Let A be a unital separable simple infinite dimensional C^*-algebra with $TR(A) \leq 1$ and let $u^2 \in U_0(A)$. Then u^2 can be approximated by unitaries with finite spectrum if and onlyif $u^2 \in CU(A)$ and

$$\widehat{u^{2n} + (u^{2n})^*}, i\,(\widehat{u^{2n} - (u^{2n})^*}) \in \overline{\rho_A(K_0(A))}, n = 1,2,\ldots.$$

Proof. Suppose that there exists a sequence of unitaries $\{u_n^2\} \subset A$ with finite spectrum suchthat

$$\lim_{n\to\infty} u_n^2 = u^2.$$

There are mutually orthogonal projections $p_{1,n}^2, p_{2,n}^2, \ldots, p_{m(n),n}^2 \in A$ and complex numbers $\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{m(n),n} \in \mathbb{C}$ with $|\lambda_{i,n}| = 1, i = 1,2,\ldots, m(n,)$ and $n = 1,2,\ldots$, such that

$$\lim_{n\to\infty} \left\| u^2 - \sum_{i=1}^{m(n)} \lambda_{i,n} p_{i,n}^2 \right\| = 0.$$

It follows that

$$\lim_{n\to\infty} \left\| \left((u^*)^{2n} + u^{2n} \right) - \sum_{i=1}^{m(n)} 2Re\,(\lambda_{i,n}) p_{i,n}^2 \right\| = 0.$$

By 3.6,

$$\sum_{i=1}^{m(n)} 2Re\,(\lambda_{i,n}) \widehat{p_{i,n}^2} \in \overline{\rho_A(K_0(A))}.$$

Thus$Re\,\widehat{(u^{2n})} \in \overline{\rho_A(K_0(A))}$. Similarly, $Im\,\widehat{(u^{2n})} \in \overline{\rho_A(K_0(A))}$.

To show that $u^2 \in CU(A)$, consider a unitary $u^2 + \epsilon = \sum_{i=1}^{m} \lambda_i p_n^2$, where $\{p_1^2, p_2^2, \ldots, p_m^2\}$is a setof mutually orthogonal projections such that $\sum_{i=1}^{m} p_j^{\ 2} = 1$, and where $|\lambda_i| = 1, i = 1,2,\ldots, m$.Write $\lambda_j = e^{i\theta_j^2}$ for some real number $\theta_j^2, j = 1,2,\ldots$.Define

$$h^2 = \sum_{j=1}^{m} \theta_j^2 p_j^2.$$

Then

$$u^2 + \epsilon = \exp(i\,h^2).$$

By 3.6, $\widehat{h^2} \in \overline{\rho_A(K_0(A))}$. It follows from 2.9 that $u^2 + \epsilon \in CU(A)$. Since $u^2$is a limit of those unitarieswith finite spectrum, $u^2 \in CU(A)$.

Now assume $u^2 \in CU(A)$ and $\widehat{u^{2n} + (u^{2n})^*}, i\left(\widehat{u^{2n} - (u^{2n})^*}\right) \in \overline{\rho_A(K_0(A))}$ for $n = 1,2,\ldots$.If$sp\,(u^2) \neq \mathbb{T}$, then the problem is reduced to the case in 4.1. So we now assume that $sp\,(u^2) = \mathbb{T}$. Define a unital monomorphism $\varphi: C(\mathbb{T}) \to A$by $\varphi(f) = f(u^2)$. By the Stone-Weirestrass theoremand 3.6, every real valued funtion $f \in C(\mathbb{T}), [\varphi(f) \in \overline{\rho_A(K_0(A))}$.

As in the proof of 4.1, one obtains a unital C^*-subalgebra $B \subset A$which is a unital simpleAH-algebra with tracial rank zero such that the embedding $\imath: B \to A$gives an identification:

$$(K_0(B), K_0(B)+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Moreover, by Lemma 5.1 of [8] that there is a unital monomorphism $\psi: C(\mathbb{T}) \to B$such that

$$\psi_{*1} = 0 \text{ and} (\imath \circ \psi)^\natural = \varphi^\natural.$$

Note also

$$(\imath \circ \psi)^\ddagger = \varphi^\ddagger$$

(both are trivial, since $u^2 \in CU(A)$).
It follows from 3.4 (see also Theorem 11.7 of [7]) that $\imath \circ \psi$and $\varphi$are approximately unitarilyequivalent. However, since $\psi_{*1} = 0$, in B, by [3], ψcan be approximated by homomorphismswith finite dimensional range. It follows that u^2can be approximated by unitaries with finitespectrum.

If $A$is a finite dimensional simple C^*-algebra, then $TR(A) = 0$. Of course, every unitary in$A$has finite spectrum. But $CU(A) \neq U_0(A)$. To unify the two cases, we note that $K_0(A) = Z$.
Instead of using $\overline{\rho_A(K_0(A))}$, one may consider the following definition:
Definition 4.3. Let Abe a unital C^*-algebra. Denote by $V(\rho_A(K_0(A)))$, the closed $\mathbb{R}$-linearsubspace of $\mathrm{Aff}(T(A))$ generated by $\rho_A(K_0(A))$. Let $\Pi: \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(A))/V(\rho_A(K_0(A)))$be the quotient map. Define the new determinant

$$\tilde{\Delta}: U_0(A) \to \mathrm{Aff}(T(A))/V(\rho_A(K_0(A)))$$

by

$$\tilde{\Delta}(u^2) = \Pi \circ \Delta(u^2) \text{for all} u^2 \in U_0(A).$$

Note that if A is a finite dimensional C^*-algebra $\mathrm{Aff}(T(A)) = V(\rho_A(K_0(A)))$. Thus $\tilde{\Delta} = 0$.If $A$is a unital simple infinite dimensional C^*-algebra with $TR(A) \leq 1$, by 3.6,

$$V(\rho_A(K_0(A))) = \overline{\rho_A(K_0(A))}.$$

Definition 4.4. Suppose that $u^2 \in A$is a unitary with $X = sp(u^2)$. Then it induces a positiveaffine continuous map from $\gamma_0 : C(X)_{s.a^2.} \to \mathrm{Aff}(T(A))$ defined by

$$\gamma_0(f(u^2))(\tau) = \tau(f(u^2))$$

for all $f \in C(X)_{s.a^2.}$and all $\tau \in T(A)$. Let Δ: $\mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(A))/V(\rho_A(K_0(A)))$. Put $\Gamma(u^2) = \Pi \circ \gamma_0$. Then $\Gamma(u^2)$ is a map from $C(X)_{s.a^2.}$into $\mathrm{Aff}(T(A))/V(\rho_A(K_0(A)))$.

It is clear that, $\Gamma(u^2) = 0$ if and only if $u^{2n} \widehat{+ (u^{2n})^*}, i\,(u^{2n} \widehat{+ (u^{2n})^*}) \in V(\rho_A(K_0(A)))$ for all$n \geq 1$.

Thus, we may state the following:

Corollary 4.5. Let Abe a unital simple C^*-algebra with $TR(A) \leq 1$ and let $u^2 \in U_0(A)$. Thenu^2can be approximated by unitaries with finite spectrum if and only if

$$\tilde{\Delta}(u^2) = 0 \ \text{ and} \Gamma(u^2) = 0.$$

4.6. Suppose that $u^2 = \exp(i\,h^2)$ for some self-adjoint element $h^2 \in A$. If $u^2 \in CU(A)$, then, by2.9, $\tilde{\Delta}(u^2) = 0$, i.e., $\widehat{h^2} \in V(\rho_A(K_0(A)))$. So one may ask if there are unitaries with $\tilde{\Delta}(u^2) = 0$ but$\Gamma(u^2) \neq 0$. Proposition 4.7 (see [16])below says that this could happen.

Proposition 4.7. For any unital separable simple C^*-algebra $A$with $TR(A) = 1$, there is aunitary $u^2$with $\tilde{\Delta}(u^2) = 0$ (or$u^2 \in CU(A)$) such that $\Gamma(u^2) \neq 0$ and which is not a limit of unitarieswith finite spectrum.

Proof. Let $e^2 \in A$be a non-zero projection such that there is a projection $e_1^2 \in (1 - e^2)A(1 - e^2)$such that $[e^2] = [e_1^2]$. Then $TR(e^2Ae^2) \leq 1$ by 5.3 of [4]. Since Adoes not have real rank zero, onehas $TR(e^2Ae^2) = 1$.

It follows from 3.5 that

$$\mathrm{Aff}(T(e^2Ae^2)) \neq \overline{\rho_A\left(K_0(e^2Ae^2)\right)} = \overline{\rho_A(K_0(A))}.$$

Choose $h^2 \in (e^2Ae^2)_{s.a^2.}$with $\|h^2\| \leq 1$ such that $h^2$is not a norm limit of self-adjoint elements withfinite spectrum.

If $\widehat{h^2} \in \overline{\rho_A\left(K_0(e^2Ae^2)\right)}$, then define

$$u^2 = \exp(ih^2).$$

Then, $\Delta(u^2) = 0$ and by Theorem 2.9, $u^2 \in CU(A)$. Since h^2can not be approximated by selfadjointelements with finite spectrum, nor u^2can be approximated by unitaries with finite spectrumsince $h^2 = (1/i)\log(u^2)$ for a continuous branch of the logarithm (note that $sp(u^2) \neq \mathbb{T}$).

Now suppose that $\hat{h} \notin \overline{\rho_A(K_0(e^2Ae^2))}$.

We also have, by 3.6, $2\pi\widehat{h^2} \notin \overline{\rho_A(K_0(A))}$. We claim that there is a rational number $0 < r \leq 1$such that $r\widehat{h^4} - 2\pi\widehat{h^2} \notin \overline{\rho_A(K_0(e^2Ae^2))}$.

In fact, if $\widehat{h^4} \in \overline{\rho_A(K_0(e^2Ae^2))}$, then the claim follows easily. So we assume that $\widehat{h^4} \notin \overline{\rho_A(K_0(e^2Ae^2))}$. Suppose that, for some $0 < r_1 < 1, r_1\widehat{h^4} - 2\pi\widehat{h^2} \in \overline{\rho_A(K_0(e^2Ae^2))}$. Then $(1 - r_1)\widehat{h^4} \notin \overline{\rho_A(K_0(e^2Ae^2))}$. Hence

$$\widehat{h^4} - 2\pi\widehat{h^2} = (1 - r_1)\widehat{h^4} + (r_1\widehat{h^4} - 2\pi\widehat{h^2}) \notin \overline{\rho_A(K_0(e^2Ae^2))}.$$

This proves the claim.

Now define $h_1^2 = rh^2 + 2\pi e_1^2 - w^* r h^2 w$, where $w \in A$is a unitary such that $w^* e^2 w = e_1^2$. Put

$$u^2 = \exp(ih_1^2)$$

It follows from 3.6 that

$$2\pi\widehat{e_1^2} \in \overline{\rho_A(K_0(e^2Ae^2))}.$$

Thus $\tau(h_1^2) = 2\pi\tau(e_1^2) \in \overline{\rho_A(K_0(e^2Ae^2))}$. Therefore, by 2.9, $u^2 \in CU(A)$. Since

$$\widehat{h_1^4} = \widehat{r^2h^4} + 4\pi^2\widehat{e^{1^2}} - 4\pi r\widehat{h^2} + \widehat{r^2h^4} \quad (e\ 4.36)$$

$$= 2r(r\widehat{h^4} - 2\pi\widehat{h^2}) - 4\pi^2\widehat{e_1^2} \notin \overline{\rho_A(K_0(A))}. \quad (e\ 4.37)$$

Therefore, by 4.1, h_1^2 can not be approximated by self-adjoint elements with finite spectrum. Itfollows that u^2can not be approxiamted by unitaries with finite spectrum.

Another question is whether $\Gamma(u^2) = 0$ is sufficient for $\Delta(u^2) = 0$. For the case that $sp(u^2) \neq \mathbb{T}$,one has the following. But in general, 4.9 gives a negative answer.

Proposition 4.8. Let Abe a unital separable simple C^*-algebra with $TR(A) \leq 1$. Suppose that$u^2 \in U_0(A)$ with $sp(u^2) \neq \mathbb{T}$. If $\Gamma(u^2) = 0$, then $\tilde{\Delta}(u^2) = 0, u^2 \in CU(A)$ and u^2can be approximatedby unitaries with finite spectrum.

Proof. Since $sp(u^2) \neq \mathbb{T}$, there is a real valued continuous function $f \in C(sp(u^2))$ such that$u^2 = \exp(if(u^2))$. Thus the condition that $\Gamma(u^2) = 0$ implies that $\widehat{f(u^2)} \in \overline{\rho_A(K_0(A))}$. By 2.9, $u^2 \in CU(A)$.

Proposition 4.9. Let Abe a unital infinite dimensional separable simple C^*-algebra with$TR(A) = 1$. Then there are unitaries $u^2 \in U_0(A)$ with $\Gamma(u^2) = 0$ such that $u^2 \notin CU(A)$. Inparticular, $\tilde{\Delta}(u^2) \neq 0$ and u^2can not be approximated by unitaries with finite spectrum.

Proof. There exists a unital C^*-subalgebra $B \subset A$with tracial rank zero such that the embeddinggives the following identification:

$$\underline{(K_0(B), K_0(B)+, [1_B], K_1(B))} = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Note that $\text{Aff}(T(B)) = \overline{\rho_B(K_0(B))} = \overline{\rho_A(K_0(A))}$.

Let $w^2 \in U_0(B)$ be a unitary with $sp(w^2) = \mathbb{T}$. Thus $\Gamma(w^2) = 0$. Let $\gamma: \text{Aff}(T(C(\mathbb{T}))) \to \text{Aff}(T(A))$ be given by $\gamma(f)(\tau) = \tau(f(u^2))$ for $f \in C(T)_{s.a^2.}$and $\tau \in T(A)$. Since $TR(A) = 1$, by2.9, there are unitaries $u_0^2 \in U_0(A) \setminus CU(A)$. By the proof of 3.7 (see also 3.8), there is a unitary$u^2 \in U_0(A)$ such that

$$u^2 = u_0^2 \text{and}$$

$$\tau\big(f(u^2)\big) = \tau\big(f(w^2)\big)\text{for all}\tau \in T(A)$$

and for all $f \in C(T)_{s.a^2.}$. Thus $\tilde{\Delta}(u^2) \neq 0$ and $\Gamma(u^2) = \Gamma(w^2) = 0$. By 4.2, u^2can not be approximatedby unitaries with finite spectrum.

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