Applications of Ordinary Differential Equations in Maple

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ABSTRACT: *In this paper, we introduce some applications of ordinary differential equations (ODEs) in Maple. Using this Maple system, we investigate the computer algebra system that can work for ODEs and development studies in Mathematics.*
KEYWORDS: *Maple system, Computer algebra system, Differential equations.*

I.   INTRODUCTION

A differential equation is an equation that involves derivatives of one or more unknown functions. Solving the differential equation means finding a function (or every such function) that satisfies the differential equation. In many models, we will have equations involving the derivatives of a dependent variable y with respect to one or more independent variables and are interested in discovering this function y. Such equations are referred to as differential equations (abbreviated DE). They arise in many applications such as population growth, decay of radioactive substance, the motion of a falling object, electrical network, and many more models.

In this paper, we define some notation to clear understand, with the following units.

mass	slugs (pound/gram),
distance	Feet,
time	Seconds,
acceleration due to gravity (g)	$32\, ft/s^2$,
position(in m)	$y(t)$,
velocity	$v(t) = \dot{y}(t) = \dfrac{d}{dt}\, y(t),$
resistant force	$f_r,$
linear drag	$f_r = -a\, v(t).$

we use the following Maple commands.

diff,	*D,*	*dsolve,*	*simplify,*	*rhs,*	*subs,*	*plot*

II. FREE FALL WITHOUT FRICTION

A bomber airplane is cruising at an altitude of 10,000 feet and drops a bomb to the target on the ground. The bomb weighs 152 pounds. Ignore its horizontal motion at the moment it becomes free of the plane. Assume that air resistance (denoted by $k\,v$, where k is a constant) is proportional to the velocity at any time *t*. The first-order ODE which describes the motion of the bomb is

$$\frac{152}{32}\frac{\partial v}{\partial t} = 152 - k\,v .$$

Now we consider how computer algebra system can work for this motion.
> *k := 'k';*

$$k := k$$

>*deqv := (152*(1/32))*(diff(v(t), t)) = 152-k*v(t);*

$$deqv := \frac{19}{4}\frac{d}{dt}v(t) = 152 - k\,v(t)$$

Since the downward motion of the bomb with an initial velocity of zero, the initial condition of $v(0) = 0$ holds.
For the velocity as a function of time,
>*solv := simplify(dsolve({deqv, v(0) = 0}, v(t)));*

$$solv := v(t) = \frac{152 - 152e^{-\frac{4}{19}kt}}{k}$$

We can now define the first-order ODE which describes the rate of change of distance with respect to time
>*deqx := diff(x(t), t) = rhs(solv);*

$$deqx := \frac{d}{dt}x(t) = \frac{152 - 152\,e^{-\frac{4}{19}kt}}{k}$$

and solve for distance as a function of time. Downward motion is used as positive distance, and the point where the fall began as the origin, such that $v(0) = 0$.
>*solx := simplify(dsolve({deqx, x(0)=0}, x(t)));*

$$solx := x(t) = \frac{152kt - 722\,e^{-\frac{4}{19}kt} - 722}{k^2}$$

We know that the bomb will be on the ground when $x(t) = 10{,}000$. The solution depends upon the value of k.
Solve for t using Maple's *fsolve* command with k equal to 0.1,1,2,3,4,5 and 6.
>*'t0.1' := fsolve(rhs(subs(k=0.1, solx)) = 10000, t);*

$$t0.1 := 27.39921852$$

This process continuous for $k = 1, 2, 3, 4, 5$ and 6, then we get

$$t1 := 70.53947200$$
$$t2 := 133.9539474$$
$$t3 := 198.9517544$$
$$t4 := 264.3453947$$
$$t5 := 329.8973684$$
$$t6 := 395.5285088 \, .$$

In this case, the large value of k (greater air resistance) will takes longer for the bomb to reach the ground. Now graph *solx* over the first 400 seconds for several different values of k.
>*curves := seq(subs(k=i, rhs(solx)), i = [0.1, 1, 2,3,4,5, 6]) :*
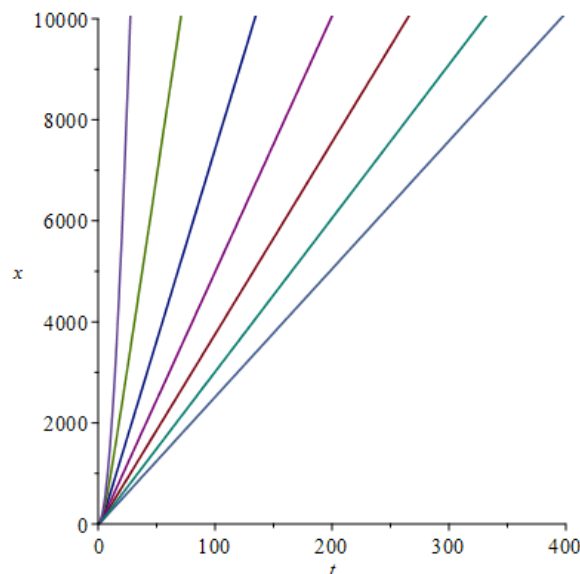>*plot({curves}, t = 0 .. 400, x = 0 .. 10000)*

Figure 1. Solution curves for several different values of k

We see that all these graphs appear linear away from the origin. Now we look at them very shortest distance and time near the origin.
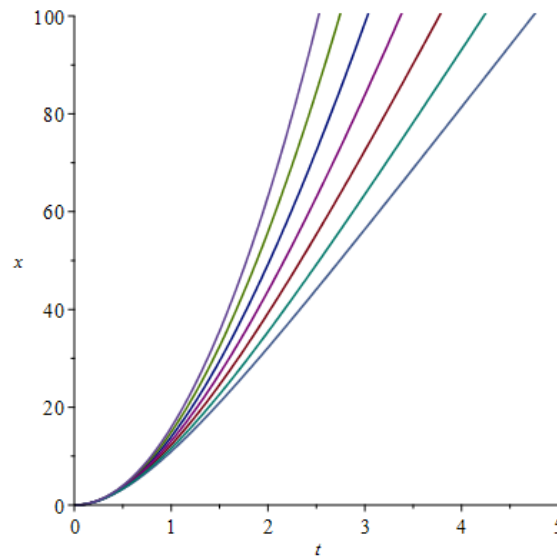
> *plot({curves}, t = 0 .. 5, x = 0 .. 100);*

Figure 2. Solution curves for shortest distance and time

This graph is very satisfied view to investigate the motion of the object near the beginning of the fall. Thus, it is apparent that for smaller values of *k* the graph of *x(t)* gets steeper faster. This means that near the beginning of the fall the bomb accelerates faster (i.e., the velocity is increasing at a quicker pace) for smaller values of *k*. Later in the fall, the velocities for all values of *k* are nearly constant (the graphs of *x(t)* appear linear for large values of *t),* with the greater limiting velocities occurring for smaller values of *k.*

III. FREE FALL WITH FRICITION

For the problem with friction drag, we can exact symbolic solution.

> *restart;*

Differential equation

>*ode := m*diff(y(t), t, t)) = -mg-a*(diff(y(t), t));*

$$ode := m\left(\frac{d^2}{dt^2}\,y(t)\right) = -mg - a\left(\frac{d}{dt}\,y(t)\right)$$

Symbolic initial condition

>*ics := y(0) = y[0], (D(y))(0) = v[0];*

$$ics := y(0) = 0,\ D(y)(0) = v_0$$

Symbolic ODE solution

>*sol := dsolve({ics, ode});*

$$sol := y(t) = -\frac{me^{-\frac{at}{m}}\left(a\,v_0 + mg\right)}{a^2} - \frac{mgt}{a} + \frac{a^2 y_0 + a\,mv_0 + m^2 g}{a^2}$$

Expression for displacement

>*fy := eval(y(t), s)*

$$fy := -\frac{me^{-\frac{at}{m}}\left(a\,v_0 + mg\right)}{a^2} - \frac{mgt}{a} + \frac{a^2 y_0 + a\,mv_0 + m^2 g}{a^2}$$

For the velocity as a function of time,

>*fv := diff(fy, t)*

$$fv := -\frac{me^{-\frac{at}{m}}\left(a\,v_0 + mg\right)}{a^2} - \frac{mg}{a}$$

Displacement for particular values

>*pvals := [m = 150, v[0] = 15, g = 32, y[0] = 5000];*

$$pvals := [m = 150, v[0] = 15, g = 32, y[0] = 5000]$$

>*fyp := eval(fy, pvals)*

$$fyp := -\frac{150\, e^{-\frac{1}{150}at}(15a+4800)}{a^2} - \frac{4800t}{a} + \frac{5000a^2+2250a+720000}{a^2}$$

When we compute the limit of *fyp* for $a \rightarrow 0$, the non-friction formula results is
>*limit(fyp, a = 0);*

$$-16t^2 + 15t + 5000$$

Velocity expression for the particular values
>*fvp := eval(fv, pvals);*

$$fvp := \frac{e^{-\frac{1}{150}at}(15a+4800)}{a} - \frac{4800}{a}$$

For the non-friction case, the limit of the velocity as a approaches 0
>*limit(fv, a = 0);*

$$v_0 - gt$$

To draw the displacement and velocity graphs for several values of the friction coefficient a, we use the *animate( )* command of the library plots.
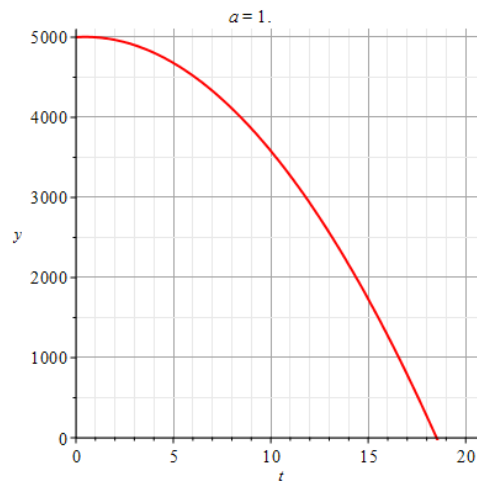>*animate(plot, [fyp, t = 0 .. 21, y = 0 .. 5100], a = 1 .. 10, gridlines, thickness = 2);*

Figure 3. The displacement graph for several values of the friction coefficient $a = 1$.

> *animate(plot, [fvp, t = 0 .. 21, v = 15 .. -600, color = blue], a = 1 .. 10, gridlines, thickness = 2);*
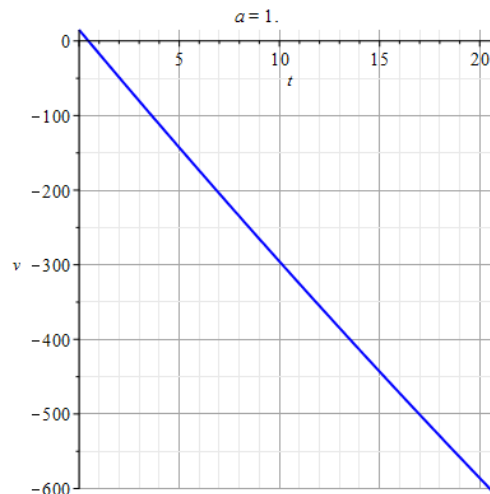
Figure 4. The velocity graph for several values of the friction coefficient $a = 1$.

> *animate(plot, [fyp, t = 0 .. 25, y = 0 .. 5100], a = 1 .. 10, gridlines, thickness = 2);*
>*animate(plot, [fvp, t = 0 .. 25, v = 15 .. -450, color = blue], a = 1 .. 10, gridlines, thickness = 2);*
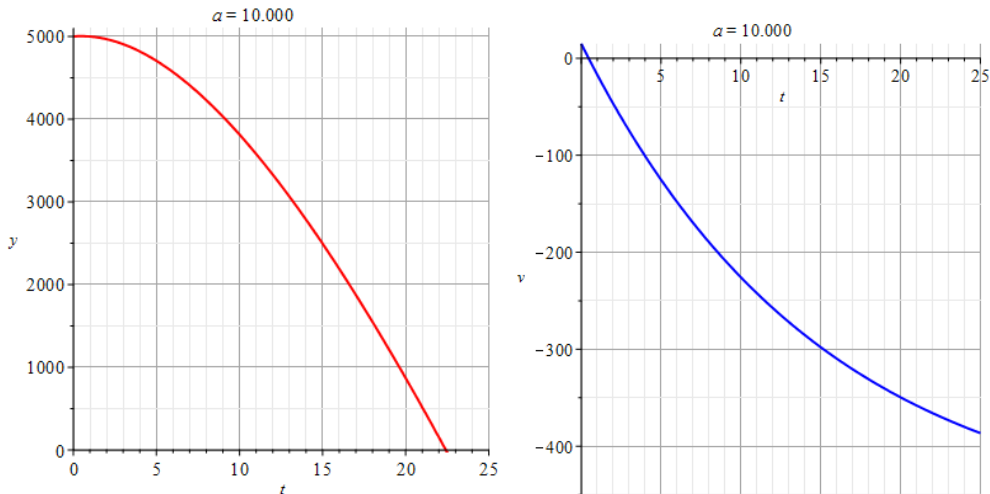

Figure 5. The displacement and velocity graphs for several values of the friction coefficient $a = 10$.

The limit velocity for each value of the linear friction coefficient a is
> *vlim := limit(fv, t = infinity);*

$$vlim := \left(\frac{e^{-\frac{at}{m}}(av_0 + gm)}{a} - \frac{gm}{a} \right)$$

Assuming the parameter *a,g* and *m* are positive,
>*vlim := `assuming`([limit(fv, t = infinity)], [m > 0, a > 0, g > 0]);*

$$vlim := -\frac{gm}{a}$$

According to the above example, the limit velocity for $a=10$ is
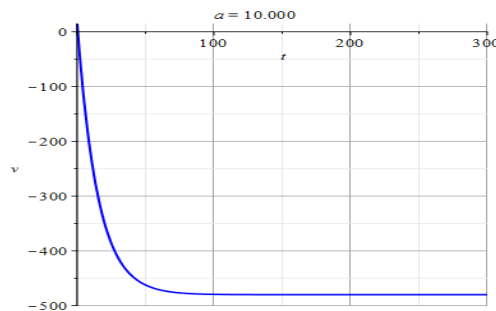>*vlima := eval(vlim, [m = 150, g = 32, a = 10]);*

$$vlima := -480$$

Figure 6. The limit velocity graph for several values of the friction coefficient $a = 10$.

For a=10, the displacement and velocity expression are
> *fypa := eval(fyp, a = 10);*
> *fvpa := eval(fvp, a = 10);*

$$fypa := -7425e^{-\frac{1}{15}t} - 480t + 12425$$

$$fvpa := 495e^{-\frac{1}{15}t} - 480$$

The time to peak is
>*tp := fsolve(fvpa = 0, t = 0 .. 25);*

$$tp := .4615748800$$

For the maximum height,

\>*ymax := eval(fypa, t = tp);*

$$ymax := 5003.444058$$

Time to impact and velocity at this moment
\>*timp := fsolve(fypa = 0, t = 0 .. 25);*

$$timp := 22.41403423$$

\>*vimp := eval(fvpa, t = timp);*

$$vimp := -368.9157621$$

IV. MIXTURE PROBLEM

A tank initially holds 40 pounds of salt dissolved in 200 gallons of water. Another brine solution is pumped into the tank at the rate of 5 gallons per minute, and a well-stirred solution simultaneously flows out of the tank atthe same rate. If the concentration of salt in the brine flowing into the tank is 3/5 pounds of salt per gallon of water, find the amount of salt present in the tank at any time *t*.
Let $A(t)$ be the amount of salt in the tank at time *t* minutes.

Then, the net rate at which $A(t)$ changes is given by

$$\frac{dA}{dt} = \text{(rate at which salt is entering)} - (\text{ rate at which salt is leaving)}.$$

The rate at which salt is entering the tank is

$$\frac{5\ gallons}{minute} \cdot \frac{3}{5} \frac{pounds}{gallons} = 3\ pounds\ per\ minute$$

The rate at which salt is leaving the tank is

$$\frac{5\ gallons}{minute} \cdot \frac{A(t)}{200} \frac{pounds}{gallons} = \frac{A(t)}{40}\ pounds\ per\ minute.$$

Thus, the first-order *ODE*s which describes this system is given by
\>*diffeq := D(A)(t) = 3-(1/40)*A(t);*

$$diffeq := D(A)(t) = 3 - \frac{1}{40}A(t)$$

The initial value of $A(t)$ is $A(0) = 40$, so we should get a unique solution from *dsolve*.
\>*sol1 := dsolve({diffeq, A(0) = 40}, A(t));*

$$sol := A(t) = 120 - 80\,e^{-\frac{1}{40}t}$$

Now we consider the amount of salt in the tank for every five minutes. In this case we set $t = 0..60$ for every five minutes.
\>*sol2 := seq(sol1, t = 0 .. 60, 5);*

$$sol2 := A(0) = 40,\qquad A(5) = 120 - 80\,e^{-\frac{1}{8}} = 49.40024779$$

$$A(10) = 120 - 80\,e^{-\frac{1}{4}} = 57.69593735,\quad A(15) = 120 - 80\,e^{-\frac{3}{8}} = 65.01685770$$

$$A(20) = 120 - 80\,e^{-\frac{1}{2}} = 71.47754722,\quad A(25) = 120 - 80\,e^{-\frac{5}{8}} = 77.17908572$$

$$A(30) = 120 - 80\,e^{-\frac{3}{4}} = 82.21067578,\quad A(35) = 120 - 80\,e^{-\frac{7}{8}} = 86.65103842$$

$$A(40) = 120 - 80\,e^{-1} = 90.56964470,\quad A(45) = 120 - 80\,e^{-\frac{9}{8}} = 94.02780261$$

$$A(50) = 120 - 80\,e^{-\frac{5}{4}} = 97.07961625,\quad A(55) = 120 - 80\,e^{-\frac{11}{8}} = 99.77283234$$

$$A(60) = 120 - 80\,e^{-\frac{3}{2}} = 102.1495872$$
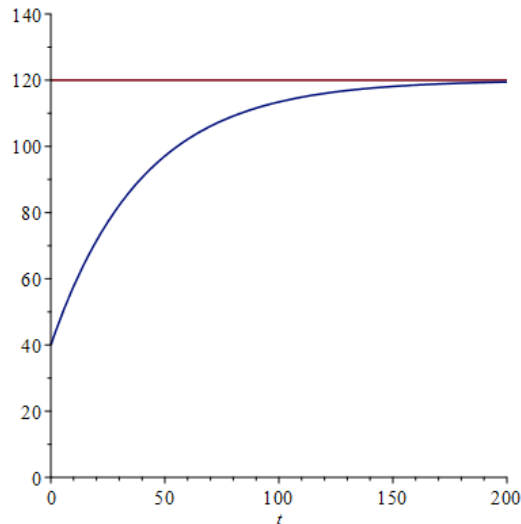
\>*plot({120, rhs(sol)}, t = 0 .. 200, 0 .. 140);*

Figure 7. Amount of salt and time

We can make a function from *sol* and use it to answer any question about the problem.
>*A :=unapply(rhs(sol), t);*

$$A := t \rightarrow 120 - 80\,e^{-\frac{1}{40}t}$$

If we want to know the amount of salt in the tank after 1 minute,
> *A(1.), 'lbs';*

41.97520704, lbs

How long does it take the amount of salt in the tank to get to 80 lbs?
>*solve(A(t) = 80, t);*

40 ln(2)

>*evalf(%), 'minutes';*

27.72588722, minutes.

V. CONCLUSION

Using trustable technique, namely, the Maple system solved ordinary differential equation with initial conditions. Maple system is a powerful tool that can use to solve mathematical problems from simple to complex. The result of this research gives that we can solve easily real life problems.

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